Closure to “Theoretical Investigation of Leak’s Impact on Normal Modes of a Water-Filled Pipe: Small to Large Leak Impedance” by Jingrong Lin, Xun Wang, and Mohamed S. Ghidaoui

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We thank the discussers for their comments on our paper and the highly meticulous and illuminating experimental results which further confirm the theory of leak’s impact on normal modes.

The aim of this closure is to more quantitatively explain the experimental results by computing the normal modes for the four groups of experiments in the discussion. Because the pipe in the Water Engineering Laboratory (WEL) of the University of Perugia is made of high-density polyethylene (HDPE) material, the viscoelasticity, as well as the steady friction, is considered in our transient model. The unsteady friction is neglected in the model, which is reasonable when the viscoelasticity is quantified (Meniconi et al. 2014). In this case, the resonant condition becomes (Wang et al. 2020)

\[
f(\omega) \triangleq \cosh(\mu(\omega)l) + \frac{Z_C(\omega)}{2Z_L} \sinh(\mu(\omega)x_L) \cosh(\mu(\omega)(l-x_L)) = 0 \quad (1)
\]

Here

\[
Z_C(\omega) = \frac{\mu(\omega)\bar{a}_e(\omega)}{i\omega gA} \quad (2)
\]

is the characteristic impedance [\(Z_C\) in the original paper is a special case of Eq. (2) where both the friction and the viscoelasticity are neglected]

\[
\mu(\omega) = \frac{\sqrt{-\omega^2 + i\omega gAR}}{a_{ve}(\omega)} \quad (3)
\]

is the propagation function, \(R\) is the steady-state resistance term being \(R = (f_{DW}Q_0)/(gDA^2)\) for turbulent flows, \(f_{DW}\) is the Darcy-Weisbach friction factor, and the wave speed is modified to be frequency dependent (Suo and Wylie 1990)

![Fig. 1. First five (normalized) normal mode frequencies in the complex plane for the four cases of leak size.](image-url)
The five FRF peaks are slightly lower than 1, 3, 5, 7, and 9, results in Fig. 9 in the discussion paper, where the locations of viscoelasticity. This shift is consistent with the experimental results in Fig. 9. In addition, it can be observed from Fig. 9 that the fourth and fifth peaks tend to emerge; this can also be explained by the theory of normal modes. As a matter of fact, in the limiting case $Z_C/Z_L \to \infty$, the system behaves as a new reservoir–pipe–valve system with no leak and with length $l - x_L = 166.493 \text{ m} - 61.3 \text{ m} = 103.63 \text{ m}$. This means that a new peak would appear at $k^* = 7.7$ in Fig. 9 and Real($\omega'$) = 7.7 in Fig. 1 (between the fourth and fifth peaks); the normalized normal mode frequencies of the new system (with length 103.63 m and no leak) are $\omega' = 1.50 + 0.05i, 4.58 + 0.13i, 7.70 + 0.23i$, and so on. This trend of normal mode shift can be found in both the analytical result in Fig. 1 and the experimental result in Fig. 9.

### Acknowledgments

This work has been supported by the Research Grant Council of the Hong Kong SAR (Project No. T21-602/15R) and the National Natural Science Foundation of China (Grant No. 52005016).

### References


### Table 1. First five (normalized) normal mode frequencies for the four cases of leak size

<table>
<thead>
<tr>
<th>Test</th>
<th>$Z_C/Z_L$</th>
<th>First mode</th>
<th>Second mode</th>
<th>Third mode</th>
<th>Fourth mode</th>
<th>Fifth mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.93 + 0.039i</td>
<td>2.85 + 0.09i</td>
<td>4.80 + 0.14i</td>
<td>6.76 + 0.20i</td>
<td>8.74 + 0.26i</td>
</tr>
<tr>
<td>2</td>
<td>0.18</td>
<td>0.93 + 0.055i</td>
<td>2.85 + 0.14i</td>
<td>4.80 + 0.14i</td>
<td>6.76 + 0.24i</td>
<td>8.74 + 0.30i</td>
</tr>
<tr>
<td>3</td>
<td>1.06</td>
<td>0.93 + 0.12i</td>
<td>2.84 + 0.37i</td>
<td>4.80 + 0.15i</td>
<td>6.76 + 0.40i</td>
<td>8.71 + 0.49i</td>
</tr>
<tr>
<td>4</td>
<td>2.96</td>
<td>0.95 + 0.26i</td>
<td>2.76 + 0.88i</td>
<td>4.79 + 0.18i</td>
<td>6.81 + 0.76i</td>
<td>8.60 + 0.88i</td>
</tr>
</tbody>
</table>

\[
a_{ee}(\omega) = \left( \rho \left( \frac{1}{\kappa} + (1 - \nu^2) \right) \frac{D}{\varepsilon} \left( J_0 + \sum_{j=1}^{N_{kv}} J_j \left( 1 + i \omega \tau_j \right) \right) \right)^{-1/2}
\]

in which $\kappa$ and $\rho$ = bulk modulus and density of water; $\nu$ = Poisson’s ratio of pipe material; $\varepsilon$ = pipe thickness; $N_{kv}$, $J_j$, and $\tau_j$ ($j = 1, \ldots, N_{kv}$) = coefficients in the generalized Kelvin-Voigt model describing the pipe viscoelasticity.

Here, we compute the normal mode frequencies, i.e., the zeros of Eq. (1), and then normalize them by the fundamental frequency $\omega_{ih} = \pi a/(2l)$, where $a$ is the elastic wave speed (361 m/s in the experiments), for the four cases of leak. The results are shown in Fig. 1 and Table 1. In the cases of no leak and small leak, the real part of each mode is slightly smaller than the corresponding elastic case ($\omega' = \omega/\omega_{ih} = 1, 3, 5, 7, and 9$) due to the presence of viscoelasticity. This shift is consistent with the experimental results in Fig. 9 in the discussion paper, where the locations of the five FRF peaks are slightly lower than 1, 3, 5, 7, and 9, respectively. Furthermore, for a large leak (e.g., $Z_C/Z_L = 2.96$), the imaginary parts of the second, fourth, and fifth modes are much larger than the other two modes, especially the third mode, which does not shift vertically in Fig. 1. This implies a larger damping in the second, fourth, and fifth peaks of FRF and almost no damping due to leak in the third peak, which is corroborated by the experimental results in Fig. 9. In addition, it can be observed from Fig. 9 that the fourth and fifth peaks tend to emerge; this can also be explained by the theory of normal modes. As a matter of fact, in the limiting case $Z_C/Z_L \to \infty$, the system behaves as a new reservoir–pipe–valve system with no leak and with length $l - x_L = 166.493 \text{ m} - 61.3 \text{ m} = 103.63 \text{ m}$. This means that a new peak would appear at $k^* = 7.7$ in Fig. 9 and Real($\omega'$) = 7.7 in Fig. 1 (between the fourth and fifth peaks); the normalized normal mode frequencies of the new system (with length 103.63 m and no leak) are $\omega' = 1.50 + 0.05i, 4.58 + 0.13i, 7.70 + 0.23i$, and so on. This trend of normal mode shift can be found in both the analytical result in Fig. 1 and the experimental result in Fig. 9.