Linear approximation of underwater sound speed profile: Precision analysis in direct and inverse problems

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**ABSTRACT**

Sound speed is one of the key sources of uncertainty in an underwater environment. The wave equation for a complex sound speed profile (SSP) cannot be analytically solved and numerical solutions generally involve high computational costs. A relatively simple way is to approximate the SSP by assuming it as horizontally stratified, vertically multi-layered and linearly varied with respect to water depth in each layer. This approximative model leads to a fast computation of the sound propagation. However, the error of SSP results in an inaccurate sound field computation, thus the model accuracy in terms of error propagation in both direct and inverse problems should be investigated. Rapidly developing numerical techniques are currently able to accurately simulate the sound propagation in a complex configuration, such that the difference between a real case with a complex SSP and its approximation can be precisely quantified. In this paper, the sound propagation with a complex SSP is simulated via a full wave numerical approach, known as the spectral element method. The efficiency of SSP linear approximation with various layer number (corresponding to different sound speed error) is quantified via transmission loss forecast (direct problem) and sound source localization error (inverse problem), respectively. The precision analysis is able to guide the choice of optimal approximate model for different scenarios, which is a trade-off between the computational cost and the model accuracy.

1. Introduction

The sound field forecast and source localization in underwater environments are challenging problems due to the complexity of the sound propagation medium [1–3]. The complex nature of sound speed is one of the main reasons, which may be inhomogeneous and imprecisely measured [4–9]. Although in many cases the water column and the seabed can be reasonably assumed as horizontally stratified, the sound speed in the vertical direction typically varies in a broad range, affected by temperature, salinity, etc. In order to simulate the sound propagation in a realistic medium with a complex sound speed profile (SSP), numerical methods, such as finite-difference, boundary-element, classical low-order finite-element methods or spectral-element method, can be used [2,10]. However, simulating a high frequency wave in a long range involves a high computational cost. In particular, source localization problems usually require a vast number of sound propagation computations. Therefore, a rapid sound field computation method is desirable.

In order to simplify the sound propagation model, a linearly varied SSP can be considered. Planar multi-layered medium is considered in Refs. [11–14] where a general solution of the depth-dependent wave equation can be given. The Helmholtz equation can be computed using the wavenumber integration [11,12] and normal mode [15,16,13] methods. However, the approximation of SSP results in an error of sound field computation which needs to be evaluated.

In this article, the underwater region is assumed to be planar multi-layered, i.e., horizontally stratified and vertically divided into multiple layers. In addition, in each layer the sound speed is constant or pseudo-linear (the square of the wavenumber linearly varies) with respect to the water depth. The sound field is computed via the wavenumber integration method [11,2]. The efficiency of this sound speed approximation to a realistic case (with a complex SSP) in the sense of sound field forecast and source localization accuracy is quantified. For this purpose, the results obtained from the multi-layer linear sound speed model is compared with a realistic SSP model; the latter is realized via a time domain full wave numerical method, known as the spectral-element method (SEM) [17,18]. This approach is a formulation of the finite-element method that uses high-degree polynomials as elemental basis functions. The accuracy of this numerical method for sound wave simulation in a complex underwater environment, with complicated

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https://doi.org/10.1016/j.apacoust.2018.05.003
Received 7 December 2017; Received in revised form 9 March 2018; Accepted 2 May 2018
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SSP, different sediment, irregular layer interface and reflections, etc., has been validated [10]. In the present paper, the considered underwater region is in 3D but the 2D version of SEM can be used since the wave equation is assumed to be independent of horizontal azimuth. More specifically, the time-domain sound wave is simulated via the open-source code SPECSEM2D based on an axisymmetric formulation [19].

In ocean acoustics, the computation error in both direct and inverse problems are concerned. The former means the forecast of sound field: it can be quantified by the transmission loss, which describes the accumulated decrease of sound energy as a wave propagates outwards from a source. The latter implies the accuracy of sound source localization. In this work, the matched-field processing (MFP) method [20–23] is used for the source localization problem. This spectral-based signal processing technique matches the field distribution versus the source location parameter. Conventional MFP [24,25] maximizes the power output of a point source and leads to a maximum likelihood estimate (MLE) of the source location and strength [26–28]. Minimum variance distortionless filter (MVDF), or Capon’s, MFP [29,30] minimizes the variance at the output of a linear weighting of the sensors subject to unit gain. Compared to the conventional MFP, the MVDF approach results in a super-resolution estimation and compresses local maxima of objective function, but is more sensitive to model uncertainties [20,31]. Furthermore, it requires a large number of data since the sample covariance matrix has to be computed while the conventional MFP works even if only one snapshot is available. In this paper, the errors of transmission loss forecast and MFP source localization are both computed, using the linearized multi-layer model with various number of layers. Actually, as the number of layers increases, both errors decrease since the sound speed is better fitted, but the computational cost increases as well. The model precision analysis in this paper provides a guidance for choosing an optimal model in terms of layer number, which is a trade-off between the computational cost and error.

The organization of the paper is as follows. Section 2 introduces the sound propagation computation with a multi-layer linear SSP, which is used to approximate a realistic underwater medium. In Section 3, the approximation efficiency of the linearized SSP is quantified in both direct and inverse problems via sound field forecast and source localization respectively. Then, a numerical example of summer Mediterranean with a complex SSP is considered in Section 4. The SEM, which is used for simulating wave propagation in the complex medium, is introduced. The errors in both the direct and inverse problems due to the linear approximation of SSP are analyzed. Finally, conclusions are drawn in Section 5.

2. Sound propagation in underwater with a multi-layer linear sound speed profile

In this section, the sound propagation model assuming multi-layer linear SSP is considered. The underwater region \( r = (x,y,z); z \in [d_0,d_L] \), where \( d_0 = 0 \) and \( d_L \) are respectively stand for the water surface and the water bottom, is horizontally stratified and divided into \( L \) layers with interfaces \( d_1, \ldots, d_{L-1} \), as shown in Fig. 1. The sound source \( r_0 = (x_0,y_0,z_0) \) is assumed to be in the \( s \)-th layer, i.e., \( x_0 \in [d_{s-1},d_s], s \in \{1, \ldots, L\} \). The sound speed in the water column is continuous with respect to the depth \( z \) and is pseudo-linear (the square of the wavenumber is varied linearly with respect to the depth) in each layer. The sound speed at each interface is

\[
v(d_i) = v_{0d} = 0.1, \ldots, L
\]

\( v_{l(z)} = \sqrt{\frac{\rho(z)}{\rho_{l-1}(d_{l-1})}}z \in [d_{l-1},d_l], \)

and in the \( l \)-layer is thus

\[
\text{in which}\]

\[
\frac{a_lz}{l^2} + b_l, z \in [d_{l-1},d_l],
\]

\]

The seabed is represented by an infinite flat halfspace \( \{r = (x,y,z); z \in [d,\infty)\} \), in which the sound speed is constant and represented by \( v_{L+1} \). The density in the water column and the ocean bottom is assumed to be constant:

\[
\rho(z) = \begin{cases} \rho_1, & z \in [0,d] \\ \rho_2, & z \in (d,\infty) \end{cases}
\]

The wave equation for the displacement potential as a function of the spatial coordinate \( r = (x,y,z) \) and time \( t \) is governed by

\[
\nabla^2 \phi + \frac{k^2}{\rho(z)} \phi = 0
\]

in which \( S_f \) is a deterministic function of source signal in the time domain, and \( \delta \) is the Dirac delta function. Taking a Fourier transform of both sides of Eq. (5) with respect to \( t \) results in the Helmholtz equation for the sound field \( \psi(x_f) \) in the frequency domain:

\[
(\nabla^2 + k^2(z))\psi(x_f) = \delta(r - r_0)S_f
\]

where \( f \) is the frequency and \( S_f \) is the Fourier transform of \( S_t \). The sound pressure is obtained from the displacement potential as

\[
p(r_f) = \rho_0\omega^2 \psi(x_f)
\]

where \( \omega = 2\pi f \) is the angular frequency.

Assuming that the sound source is omni-directional, thus the sound field only depends on the depth and the horizontal range. Then, a cylindrical coordinate system is chosen, i.e., the spatial coordinate is denoted by \( r = (r,z,\phi) \), in which \( r \) and \( \phi \) stand for the horizontal range and azimuth, respectively. By applying the Hankel transform

\[
\psi(k_z) = \int_0^\infty \psi(r,z,\phi)J_0(kr)dr
\]

to Eq. (6), the depth-separated wave equation is obtained:

\[
\frac{\beta^2}{\delta z^2} + (k^2(z)-k_0^2)\psi(k_z) = \frac{\delta(z-z_0)}{2\pi}
\]
Eq. (9) can be analytically solved, which is detailed in Appendix A. Then, the Helmholtz Eq. (6) is solved by applying an inverse Hankel transform on $\psi(k, z)$, known as the wavenumber integration method [11,2], i.e., numerically computing the onefold integration

$$\psi(r, z) = \int_{k_0}^{\infty} \psi(k, z) \delta_r(k, r) dk,$$

(10)

3. Precision analysis in direct and inverse problems due to sound speed linearization

In this section, the accuracy of linear approximation of SSP is evaluated in both direct and inverse problems, which are introduced in Sections 3.1 and 3.2 respectively.

3.1. Direct problem

Let $v^s(z)$ and $v(z)$ denote the actual and linearly approximated SSPs. The mean error reads

$$e_v = \frac{1}{d} \int_{0}^{d} |v^s(z) - v(z)| dz.$$  

(11)

The sound pressures with the actual and linearly approximated SSPs are denoted by $p^s$ and $p$, respectively. The corresponding transmission losses are obtained by computing the sound energy decrease in a line along the x-axis:

$$TL(x) = -20 \log \left( \frac{|p^s(x,y,z_2)|}{|p_1|} \right) x \in [0, \infty)$$

(12)

and

$$TL(x) = -20 \log \left( \frac{|p(x,y,z_2)|}{|p_1|} \right) x \in [0, \infty),$$

(13)

in which $y_1$ and $z_2$ are fixed, $p^s_1$ and $p_1$ are the sound pressures 1 m away from the sound source. In this paper, the approximation efficiency of SSP in the direct problem is evaluated by the mean error of transmission loss

$$e_{TL} = \frac{1}{x_c} \int_{0}^{x_c} |TL(x) - TL(x)| dx,$$

(14)

in which $x_c$ is the cut-off point in the x-axis.

3.2. Inverse problem

The approximation accuracy of SSP in the inverse problem is assessed by the source localization accuracy. In this work, the conventional and MVDF MFP methods are used as the source localization techniques, which are briefly introduced in this section.

Let $p_i = (p_{i1}, \ldots, p_{iM})^T, t = 1, \ldots, T$, denote $T$ samples of sound pressure measurements in the frequency domain (with real SSP) obtained from $M$ measurement stations, $\overrightarrow{K} = \frac{1}{T} \sum_{t=1}^{T} pp^T$ denote the sample covariance matrix and $G(r)$ be the Green’s function at the $M$ stations with linearized SSP, which is computed using the wavenumber integration method. The conventional and MVDF MFP methods [26,20] estimate the sound source location by

$$\hat{r}_{0} = \arg\max_{r_{0}} \frac{G^H(r_0) \overrightarrow{K} G(r_0)}{G(r_0)G(r_0)},$$

(15)

and

$$\hat{r}_{0} = \arg\max_{r_{0}} \frac{1}{G^H(r_0) \overrightarrow{K} G(r_0)},$$

(16)

respectively. Note that in the source localization problem, the Green’s function $G(r)$ has to be repeatedly computed. The sound speed linearization thus largely decreases the computational cost of source localization. The efficiency of sound speed approximation in the inverse problem is assessed via the source localization error $|\hat{r}_0 - r_0|$.

4. Numerical experiments

In this section, numerical examples are investigated to show the approximation efficiency of linearized SSP. The sound propagation with complex sound speed is realized by the time-domain numerical approach SEM, which is introduced in Section 4.1. Then, Sections 4.2 and 4.3 show the results for direct and inverse problems, respectively.

4.1. Sound propagation simulation with a complex sound speed profile

In this work, sound propagation simulations in an underwater environment with a complex SSP are carried out via the SEM, which is based on a high-order piecewise polynomial approximation of the weak formulation of the wave equation. It has been shown in Ref. [10] that this technique is able to accurately simulate the acoustic wave propagation in a complex underwater environment. Since the wave equation is independent of the horizontal azimuth, the 2D axisymmetric version of SEM can be used based on an axisymmetric formulation [19] implemented in the open-source code SPECFEM2D.

In this section, an example of summer Mediterranean shallow water (cf. Fig. 1.15 in Ref. [2]) is considered; the SSP in the water column is shown by the solid line in Fig. 2. The water depth is $d = 100$ m. The sound speed is in the range of $[1510,1537.5]$ m/s in the water column and is $1800$ m/s in the seabed; the density in the two areas are $1000$ kg/m$^3$ and $1800$ kg/m$^3$, respectively.

The source signal is a Ricker wavelet, i.e., the second derivative of a Gaussian function, whose dominant frequency is $f_0 = 100$ Hz. In order to guarantee the accuracy of the SEM, the length $\Delta x$ of each element in each direction has to be smaller than $\Delta x_{\min} = \Delta x_{\min} = 6$ m, in which $\Delta x_{\min} \approx \text{Vmax} / (2.5f_0)$ is the minimum wavelength. Furthermore, the CFL stability condition of the time-integration scheme implies that the time step $\Delta t$ should be smaller than $c \Delta x_{\min} / V_{1, x} = 0.0013$ s. Here, $c$ is the Courant number and is taken as $0.4$ for the SEM. In the numerical simulation introduced in this section, the element length is chosen to be $\Delta x = 1$ m; in each element 9 Gauss-Lobatto-Legendre integration points are taken in each direction (thus 81 points in total). The time step is $\Delta t = 2 \times 10^{-4}$ s, which ensures the accuracy of the wave propagation simulation. The sound source is located at $r_0 = (0, 0.50)$ m and the horizontal range in the computation is 10 km. The total simulation time is $30$ s, such that the direct propagation and the main reflections of sound wave all pass through the considered region. For the measured time signal at each position in the medium, its frequency-domain pressure is obtained by performing a discrete Fourier transform. Then, the result can be compared with the frequency-domain solution introduced in Section 2.

4.2. Direct problem: transmission loss forecast

In this section, the SSP of summer Mediterranean shallow water is approximated by several line segments. The computation accuracy in the direct problem is considered in terms of transmission loss with respect to various layer number. Note that as the layer number $L$ increases, the complex SSP is better approximated (Fig. 2 shows the cases of $L = 1$, $L = 3$ and $L = 5$) and therefore the sound field can be better estimated. However, increasing $L$ also implies higher computational costs. Fig. 3 shows the computational time under the experimental setup in Section 4.1 normalized by the case $L = 1$, which almost

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1 This code can be found here https://geodynamics.org/cig/software/specfem2d/.

2 Based on numerical simulations of acoustic wave propagation, the maximum values of Courant number $c$ are respectively recommended as 0.58 and 0.71 and 0.58 for 3D, 2D and 1D media [32,33]. Thus, $c = 0.4$ is conservative enough.
linearly increases with respect to layer number $L$. As a matter of fact, the wavenumber integration method proposed in Section 2 is efficient, i.e., a single simulation of sound propagation is fast (only a onefold numerical integration is needed). However, in the case that the simulation has to be repeatedly done, for example in the source localization problem that the sound propagation from all the possible source locations to all the measurement points has to computed, the computational cost becomes high and needs to be optimized. More specifically, a minimum layer number (an optimal model) that ensures the computation accuracy is proposed.

Here, the transmission loss at 100 Hz and at depth of 51 m is computed. The original SSP and its linear approximation with various layer number are compared. Fig. 4 illustrates that the model with single layer results in an erroneous estimation of sound field. By contrast, in the case of $L = 4$ the trend of transmission loss is approximately replicated, especially in the short ranges. As the sound propagation range
increases, the error of transmission loss increases as well, since the computation error is accumulated as the sound propagates to a longer distance in a sound field with imprecise information of SSP. Remark that Ref. [14] shows that even the 3-layer model \( L = 3 \) can be reasonably used for source localization, but it cannot precisely replicate the transmission loss in a long range. However, when the SSP is better fitted, as is shown in Fig. 4 (c) wherein \( L = 7 \), the transmission loss is well-forecast up to 10 km. Fig. 5 displays the mean error of sound speed (left) with various layer number and the corresponding mean error of transmission loss (right): the former and the latter are respectively obtained from Eqs. (11) and (14). It is clear that as the layer number increases both errors decrease. Besides, when \( L \geq 7 \) the mean error of
transmission loss is less than 3 dB, which signifies that the layer number \( L = 7 \) might be a good trade-off between the forecast precision and computational cost.

4.3. Inverse problem: source localization

In this section, the sound source localization accuracy using the model with linearly approximated SSP is investigated. Eleven towed sensor arrays are assumed to be placed, each of which includes ten receivers: their coordinates are \((x'_m, y'_m, z'_m)\), \(z'_m = -1000, -800, \ldots, 1000\) m, \(x'_m = 5, 15, \ldots, 95\) m. Various distance from the source to the measurement plane, i.e., \(|x'_m - x_s|\), is tested. The simulated measurements at the sensors are under the assumption of the complex SSP (the solid line in Fig. 2) and obtained via SPECFEM2D code. A random Gaussian noise with SNR = 20 dB is then added to the simulation result and the sample size \( T = 220 \). In the inverse problem, the source is localized using the model with linearized SSP. The conventional and MVDF MFP methods are used to localize the sound source.

Fig. 6(a) and (b) display the localization results obtained from Eqs. (15) and (16) using a single-layer sound speed model and the measurement plane is 5 km away from the source. In this case, both source localization methods do not work due to the high error of sound speed. By using a three-layer model, the source localization for this source-measurement distance is largely improved that both methods accurately estimate the source location (subfigures (c) and (d)). However, a larger measurement distance from the source leads to a higher error. Subfigures (e) and (f) show that the same model \( L = 3 \) generates a higher error in the case \(|x'_m - x_s| = 10\) km, in particular for the MVDF approach which is more sensitive to model uncertainties than the conventional MFP [20]. The result can be improved by better fitting the SSP. Fig. 6(g) and (h) justifies that the nine-layer model guarantees the source localization accuracy for both conventional and MVP methods.

The error of the localization methods with various layer number \( L = 3, 4, 6, 9 \) versus various measurement distance (from 2 km to 10 km) is shown in Fig. 7. The SNR is 20 dB in (a and b), −10 dB in (c and d) and, −15 dB in (e and f). In the latter two cases, due to high level of noise (randomness), each simulation is repeated 100 times and the average error and the 95% confidence interval of error are plotted. Note that as the measurement distance increases, which implies that the sound wave propagates further in a field with imprecise information, a higher precision of sound speed (i.e., a larger \( L \)) is needed to guarantee the same accuracy of source localization. Besides, a higher noise level

![Fig. 7. Mean square error of sound source localization using the conventional (a, c, e) and MVDF (b, d, f) MFP methods. SNR is 20 dB in (a and b), −10 dB in (c and d) and, −15 dB in (e and f).]
decreases the accuracy of leak localization, especially for MVDF which is more sensitive to noise and model uncertainty. For example, when the distance between the sound source and the measurement plane $|x_L-x_s|=10$ km, the MVDR localization with $L=3$ has a large error (larger than 10 m) which is higher than the resolution (half-wavelength $\lambda/2 = 7.5$ m) from Shannon-Nyquist sampling theorem [34,35]. A model with more layers, for example $L=9$, decreases the localization error down to around 5 m.

5. Conclusions

In order to simplify the computation of wave propagation in an underwater environment, in this paper the sound speed profile is assumed to be horizontally stratified, vertically multi-layered and linearly varied in each layer. Under this approximation assumption, sound field can be rapidly computed using an integral transform approach. The influence of the approximate sound speed is investigated in both direct and inverse problems. The former and the latter are quantified by transmission loss forecast and sound source localization, respectively. In the numerical example, the sound propagation with a complex sound speed profile is realized via the spectral element method, which is a time domain full wave numerical approach. The errors in the direct and inverse problems due to imprecise sound speed are analyzed, which justifies that the model of sound speed linearization is reasonable. The error depends on the measurement range and the signal processing techniques (in the inverse problem). An optimal model (layer number), which guarantees the model accuracy with minimum computational cost, depends actually on the complexity of sound speed profile, water depth, and error tolerance of the forward or inverse problem. However, it can be given for each scenario via the proposed methodology; the precision analysis in this paper is able to guide the choice of optimal model for different problems and accuracy requirements.

The proposed model and precision analysis method can be applied for more complicated underwater environments. For example, the robustness of the proposed model with respect to range-dependent or random sound speed model and different sediment, topography, multi-layer and non-parallel interfaces could be studied. In these cases, a larger computer may be required for the time domain full wave simulation.

Acknowledgments

This work was supported by Guangxi Key Laboratory of Manufacturing Systems and Advanced Manufacturing Technology (No. 17-259-05-002Z) and China Postdoctoral Science Foundation (No. 2018M633113). The authors would like to thank Dimitri Komatitsch (LMA, Aix-Marseille University, France) for his help on SPECEM2D.

Appendix A. Solution of depth-separated wave equation

The solution of the depth-separated wave Eq. (9) in the source layer [2] is

$$\psi_i(k,z) = -\frac{S(f_i(k,z))}{4\pi} + A_i^+Ai(\xi_i^+) + A_i^-Ci(\xi_i^-), z \in (d_{i-1}, d_i],$$

(A.1)

in which $f_i(k,z)$ is the solution of Eq. (9) in a boundary-free space and equal to

$$f_i(k,z) = \begin{cases} f_i^+(k,z) = \frac{2i(\omega^2a_k^2-\omega^2z^2)}{\omega(\xi_i^+\xi_i^-\xi_i^+\xi_i^-)} \cdot z < z_o \\ f_i^-(k,z) = \frac{2i(\omega^2a_k^2-\omega^2z^2)}{\omega(\xi_i^+\xi_i^-\xi_i^+\xi_i^-)} \cdot z > z_o \end{cases}$$

(A.2)

In this equation, $Ci = Ai - iBi, Ai$ and $Bi$ are the Airy functions of first and second kinds, $i = \sqrt{-1}$, and $\xi_i^\pm$ is a variable transformation

$$\xi_i = (\omega^2a_k^2)^{-1/2}[k^2-\omega^2(a_kz + b_k)],$$

(A.3)

and $\xi_{i\pm}^\pm$ is obtained by inserting $z_o$ into Eq. (A.3). The last two terms in Eq. (A.1) are the solutions of the homogeneous depth-separated wave equation (i.e., the right hand side of Eq. (9) is 0) and respectively stand for downgoing and upgoing waves, $A_i^+$ and $A_i^-$ are parameters to be determined via boundary conditions. In a sourceless layer ($l \neq s$), since the right hand side of Eq. (9) is equal to 0, the solution of the depth-separated wave equation becomes

$$\psi_i(k,z) = A_i^+Ai(\xi_i^+) + A_i^-Ci(\xi_i^-), z \in (d_{i-1}, d_i],$$

(A.4)

for $l = 1,\ldots,s-1, s+1,\ldots,L$. In the lower infinite halfspace, the general solution includes a single term due to no upgoing wave:

$$\psi_{l+1}(k,z) = A_{l+1}^+e^{ikz_{l+1}(z-d_i)}, z \in (d_i, \infty),$$

(A.5)

in which

$$k_{l+1} = \begin{cases} k_{l+1}^- - k_{l+1}^-, |k| \leq k_{l+1}^- \\ k_{l+1}^+ - k_{l+1}^+, |k| > k_{l+1}^+ \end{cases}$$

(A.6)

In the following, the parameters $A_i^+, A_i^-,\ldots,A_{L+1}^+, A_{L+1}^-$ are selected by considering the boundary conditions at $z = d_i, l = 0,1,\ldots,L$. First, the water surface $z = 0$ is assumed to be a pressure-free boundary, such that the pressure vanishes at $z = 0$, i.e.,

$$\psi_i(k,0) = 0.$$  

(A.7)

Then, the continuity of vertical displacement and pressure at $z = d_i, l = 1,\ldots,L$, is enforced, i.e.,

$$\frac{\partial \psi_i}{\partial z}(k,d_i) = \frac{\partial \psi_{i+1}}{\partial z}(k,d_i), l = 1,\ldots,L,$$

(A.8)

$$\psi_i(k,d_i) = \psi_{i+1}(k,d_i), l = 1,\ldots,L-1$$

(A.9)
\[ \rho_1 \psi_L(k_r,d) = \rho_1 \psi_{L+1}(k_r,d). \]  
(A.10)

By solving Eqs. (A.7)–(A.10) (the detailed computation can be found in Appendix B), the coefficient \( A = (A_1^\alpha, A_1^\beta, \ldots, A_L^\alpha, A_L^\beta)^T \) is decided by \( QA = S \).  
(A.11)

in which

\[
Q = \begin{pmatrix}
Q_0 & Q_1^+ & Q_2^+ & \cdots & Q_{L-1}^+ & Q_L^+
\end{pmatrix},
\]

\[
S = \begin{pmatrix}
S_{i-1} \\
S_i
\end{pmatrix}
\]

(A.12)

The non-zero blocks in \( Q \) are given as follows:

\[
Q_0 = \left( \text{Ai}(\xi_{d,0}), \text{Ci}(\xi_{d,0}) \right),
\]

(A.13)

\[
Q_l^+ = \begin{pmatrix}
- (\omega^2 a_l)^{1/3} \text{Ai}(\xi_{d,l}) & - (\omega^2 a_l)^{1/3} \text{Ci}(\xi_{d,l}) \\
\text{Ai}(\xi_{d,l}) & \text{Ci}(\xi_{d,l})
\end{pmatrix},
\]

(A.14)

\[
Q_l^0 = \begin{pmatrix}
\frac{\rho_1}{\rho_2} & 1 & 0 & \cdots & 0 & 0 \\
0 & \frac{\rho_1}{\rho_2} & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \frac{\rho_1}{\rho_2} & 1 \\
0 & 0 & 0 & \cdots & 0 & \frac{\rho_1}{\rho_2}
\end{pmatrix},
\]

(A.15)

for \( l = 1, \ldots, L-1 \), and

\[
Q_L^0 = \begin{pmatrix}
\frac{\rho_1}{\rho_2} & 1 & 0 & \cdots & 0 & 0 \\
0 & \frac{\rho_1}{\rho_2} & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \frac{\rho_1}{\rho_2} & 1 \\
0 & 0 & 0 & \cdots & 0 & \frac{\rho_1}{\rho_2}
\end{pmatrix},
\]

(A.16)

The two non-zero blocks in \( S \) are given by

\[
S_{i-1} = \frac{S_f}{4\pi} \left( (\omega^2 a_l)^{1/3} \frac{g_r}{\xi_{d,l}}(k_r,d_{l-1}) \right), \quad i \in \{2, \ldots, L\},
\]

(A.18)

\[
S_i = \frac{S_f}{4\pi} \left( (\omega^2 a_l)^{1/3} \frac{g_r}{\xi_{d,l}}(k_r,d) \right), \quad i \in \{1, \ldots, L-1\}.
\]

(A.19)

In the case of \( s = 1 \), i.e., the source is located in the first layer,

\[
S_0 = -\frac{S_f}{4\pi} f_0^- (k_r,0),
\]

(A.20)

if \( s = L \),

\[
S_L = \frac{S_f}{4\pi} \left( (\omega^2 a_L)^{1/3} \frac{g_r}{\xi_{d,L}}(k_r,d) \right).
\]

(A.21)

In the above presentation, the sound speed in each layer is linearly varied. For the case of constant sound speed in each layer, the corresponding results can be similarly obtained, which only changes some submatrices in \( Q \) and \( S \). The details can be found in Appendix C.

Appendix B. Derivation of Eq. (A.11)

The boundary condition Eq. (A.7) results in

\[
A_1^\alpha \text{Ai}(\xi_{d,0}) + A_1^\beta \text{Ci}(\xi_{d,0}) = g_r.
\]

(B.1)

The right hand side of Eq. (B.1) depends on if the source is in the first layer:

\[
g_r = \begin{cases}
0, & s \neq 1 \\
-\frac{S_f}{4\pi} f_0^- (k_r,0), & s = 1.
\end{cases}
\]

(B.2)

For \( s = 1, \ldots, L-1 \), the boundary condition Eqs. (A.8) and (A.9) leads to
\[ -A_l^+(\omega^2 a^2)^{1/3} \text{Ai}(\xi_0, d_l) - A_l^-(\omega^2 a^2)^{1/3} \text{Ci}(\xi_0, d_l) + A_{l+1}^+(\omega^2 a^2)^{1/3} \text{Ai}(\xi_{l+1}, d_l) + A_{l+1}^-(\omega^2 a^2)^{1/3} \text{Ci}(\xi_{l+1}, d_l) = g_2 \]  
(B.3)

and

\[ A_l^+ \text{Ai}(\xi_0, d_l) + A_l^- \text{Ci}(\xi_0, d_l) - A_{l+1}^+ \text{Ai}(\xi_{l+1}, d_l) - A_{l+1}^- \text{Ci}(\xi_{l+1}, d_l) = g_3^l \]  
(B.4)

in which

\[ g_2 = \begin{cases} 0, & s \neq l, s \neq l + 1 \\ \frac{-s}{4\pi a^2} (\omega^2 a^2)^{1/3} (k, d_l), & s = l \\ \frac{-s}{4\pi a^2} (\omega^2 a^2)^{1/3} (k, d_l), & s = l + 1 \end{cases} \]  
(B.5)

and

\[ g_3 = \begin{cases} 0, & s \neq l, s \neq l + 1 \\ -\rho_l \frac{-s}{2\pi a^2} f^l (k, d_l), & s = l \\ \frac{-s}{2\pi a^2} f^l (k, d_l), & s = l + 1 \end{cases} \]  
(B.6)

Finally, for \( s = L \), the following equations are obtained from the boundary condition Eqs. (A.8) and (A.10):

\[ -A_L^+(\omega^2 a^2)^{1/3} \text{Ai}(\xi_0, d_L) - A_L^- (\omega^2 a^2)^{1/3} \text{Ci}(\xi_0, d_L) - A_{L+1}^+ i k_{L+1} = g_4 \]  
(B.7)

and

\[ A_L^+ \rho_l \text{Ai}(\xi_0, d_L) + A_L^- \rho_l \text{Ci}(\xi_0, d_L) - A_{L+1}^+ \rho_l = g_5 \]  
(B.8)

where

\[ g_4 = \begin{cases} 0, & s \neq L \\ \frac{-s}{4\pi a^2} (\omega^2 a^2)^{1/3} (k, d_L), & s = L \end{cases} \]  
(B.9)

and

\[ g_5 = \begin{cases} 0, & s \neq L \\ -\rho_l \frac{-s}{2\pi a^2} f^L (k, d_L), & s = L \end{cases} \]  
(B.10)

Eq. (A.11) is obtained by writing Eqs. (B.1), (B.3), (B.4), (B.7) and (B.8) in the form of matrix.

Appendix C. Constant sound speed in each layer

The case of constant sound speed in each layer is considered. If the sound source is in an isospeed layer, i.e., \( v_{s-1} = v_s \), the solution of the depth-separated wave equation in this layer has the form

\[ \psi_l(k, z) = S_l e^{i \beta_{sz} (z - z_{s-1})} + A_l^+ e^{i k_z z} + A_l^- e^{-i k_z z}, z \in (d_{s-1}, d_s]. \]  
(C.1)

In a sourceless layer, if the sound speed is constant, i.e., \( v_{s-1} = v_s \), the solution of the depth-separated wave equation is

\[ \psi_l(k, z) = A_l^+ e^{i k_z z} + A_l^- e^{-i k_z z}, z \in (d_{s-1}, d_s] \]  
(C.2)

for \( l \in [1, \ldots, s - 1, s + 1, L] \). Here,

\[ k_z = \begin{cases} \sqrt{k_0^2 - k_l^2}, & k_l \leq k_l = \frac{2\pi f}{v_s} \\
\frac{2\pi f}{v_s} > k_l \end{cases} \]  
(C.3)

If \( v_0 = v_s \), the boundary condition Eq. (A.7) results in

\[ A_L^+ + A_L^- = g_s, \]  
(C.4)

in which

\[ g_s = \begin{cases} 0, & s \neq 1 \\ -\frac{-s}{4\pi a^2} & s = 1 \end{cases} \]  
(C.5)

Then, the boundary condition at \( z = d_s \) is considered. If \( v_{s-1} \neq v_s \) and \( v_t = v_{s+1} \), \( t \in [1, \ldots, L-1] \), Eqs. (A.8) and (A.9) imply

\[ -A_t^+ (\omega^2 a^2)^{1/3} \text{Ai}(\xi_{s-1}, d_t) - A_t^- (\omega^2 a^2)^{1/3} \text{Ci}(\xi_{s-1}, d_t) - A_{t+1}^+ i k_{t+1} e^{i k_{t+1} z} + A_{t+1}^- i k_{t+1} e^{-i k_{t+1} z} = g_2 \]  
(C.6)

and

\[ A_t^+ \text{Ai}(\xi_{s-1}, d_t) + A_t^- \text{Ci}(\xi_{s-1}, d_t) - A_{t+1}^+ e^{i k_{t+1} z} - A_{t+1}^- e^{-i k_{t+1} z} = g_5 \]  
(C.7)

in which
\[ g_s = \begin{cases} 0, & s \neq l, s \neq l + 1 \\ \frac{S_j}{4\pi} (\omega^2 q_i)^{1/2} \left( k_r, d_l \right), & s = l \\ -\frac{S_j}{4\pi} e^{ik_r(l+1)} (\omega^2 q_i), & s = l + 1 \end{cases} \]  

(C.8)

and

\[ g_s = \begin{cases} 0, & s \neq l, s \neq l + 1 \\ -\frac{S_j}{4\pi} f_i (k_r, d_l), & s = l \\ \frac{S_j}{4\pi} e^{ik_r(l+1)} (\omega^2 q_i), & s = l + 1 \end{cases} \]  

(C.9)

Similarly, if \( v_{L-1} = v_L \) and \( v_L \neq v_{L+1,1} \in [1, \ldots, L-1] \), Eqs. (A.8) and (A.9) lead to

\[ A_i^+ e^{ik_Ld} - A_i^- e^{-ik_Ld} + A_i^+ (\omega^2 q_i)^{1/2} A_i (\xi_{1+1,0}) + A_i^- (\omega^2 q_i)^{1/2} A_i (\xi_{1+1,0}) = g_s' \]  

(C.10)

and

\[ A_i^+ e^{ik_Ld} + A_i^- e^{-ik_Ld} - A_i^+ A_i (\xi_{1+1,0}) - A_i^+ A_i (\xi_{1+1,0}) = g_s' \]  

(C.11)

in which

\[ g_s' = \begin{cases} 0, & s \neq l, s \neq l + 1 \\ -\frac{S_j}{4\pi} e^{ik_L(d-z_0)}, & s = l \\ \frac{S_j}{4\pi} (\omega^2 q_i)^{1/2} \left( k_r, d_l \right), & s = l + 1 \end{cases} \]  

(C.12)

and

\[ g_s' = \begin{cases} 0, & s \neq l, s \neq l + 1 \\ -\frac{S_j}{4\pi} f_i (k_r, d_l), & s = l \\ \frac{S_j}{4\pi} e^{ik_r(l+1)} (\omega^2 q_i), & s = l + 1 \end{cases} \]  

(C.13)

Finally, the boundary condition at \( z = d \) is considered. If \( v_{L-1} = v_L \), Eqs. (A.8) and (A.10) result in

\[ A_i^+ e^{ik_Ld} - A_i^- e^{-ik_Ld} - A_i^+ A_i (\xi_{1+1,0}) = g_s \]  

(C.14)

and

\[ A_i^+ e^{ik_Ld} + A_i^- e^{-ik_Ld} - A_i^+ A_i (\xi_{1+1,0}) = g_s \]  

(C.15)

where

\[ g_s = \begin{cases} 0, & s \neq L \\ -\frac{S_j}{4\pi} e^{ik_L(d-z_0)}, & s = L \end{cases} \]  

(C.16)

and

\[ g_s = \begin{cases} 0, & s \neq L \\ -\frac{S_j}{4\pi} e^{ik_L(d-z_0)}, & s = L \end{cases} \]  

(C.17)

If the sound speed in each layer is constant, instead of linear varied, some of the blocks in \( Q \) and \( S \) in Eq. (A.11) are replaced according to the above computations. The detailed strategy is concluded as follows. If the \( l \)-th layer is an isospeed layer, i.e., \( v_{l-1} = v_L \), the blocks \( Q_{ii,l} \) and \( Q_{ll,l} \) are

\[ Q_{ii} = \begin{pmatrix} i k_L e^{ik_Ld} & -i k_L e^{-ik_Ld} \\ e^{ik_Ld} & e^{-ik_Ld} \end{pmatrix}, \quad \text{if } l = 1, \ldots, L-1, \]  

(C.18)

and

\[ Q_{ll} = \begin{pmatrix} -i k_L e^{ik_Ld} & i k_L e^{-ik_Ld} \\ e^{ik_Ld} & e^{-ik_Ld} \end{pmatrix}, \quad \text{if } l = 2, \ldots, L. \]  

(C.19)

Furthermore, if \( v_0 = v_L \),

\[ Q = (1,1), \]  

(C.20)

and if \( v_{L-1} = v_L \),

\[ Q_{ii} = \begin{pmatrix} i k_L e^{ik_Ld} & \rho_i e^{ik_Ld} \\ \rho_i e^{ik_Ld} & -i k_L e^{-ik_Ld} \end{pmatrix}. \]  

(C.21)

If the sound source is a isospeed layer, i.e., \( v_{l-1} = v_L \),
\[ S_{i+1} = \frac{S_i}{4\pi} - \frac{e^{ik_{2,i+1}(t_0-d_i)}}{ik_{2,i+1}} \text{if } l \in 2, \ldots, L \]  \hfill (C.22)

and

\[ S_l = -\frac{S_l}{4\pi} e^{ik_{2,l}(d_l-d_0)} \frac{e^{ik_{2,l}(d_l-d_0)}}{ik_{2,l}} \text{if } l \in 1, \ldots, L-1. \]  \hfill (C.23)

Besides, if \( s = 1 \),

\[ S_0 = -\frac{S_0}{4\pi} e^{ik_{2,0}(d_0-d_0)} \] \hfill (C.24)

and if \( s = L \),

\[ S_L = -\frac{S_L}{4\pi} e^{ik_{2,L}(d_L-d_0)} \] \hfill (C.25)

References