Effects of thermocapillarity on the dynamics of an exterior coating flow of a self-rewetting fluid

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The effects of thermocapillarity on the dynamics of an exterior coating flow of a self-rewetting fluid on a vertical fibre are investigated theoretically. Whereas surface tension decreases linearly with temperature for most fluids, the surface tension of a self-rewetting fluid exhibits a well-defined minimum. We developed an evolution equation for the interface in the framework of the long wave approximation. Linear stability analysis and numerical simulations on the nonlinear evolution have been performed to investigate the effects of thermocapillarity for axisymmetric disturbances. The results showed that thermocapillarity plays different roles in the stability and dynamics depending on the value of difference between the temperature at the interface, \(\Theta_i\), and the temperature corresponding the minimum of the surface tension, \(\Theta_0\). The results of linear stability analysis showed that the thermocapillarity is destabilizing or stabilizing as \(\Theta_i - \Theta_0\) is negative or positive. At the nonlinear stage, for \(\Theta_i - \Theta_0 < 0\) more pronounced beads-like structures are observed due to the reinforcement between the thermocapillarity and the Rayleigh-Plateau instability. However, for \(\Theta_i - \Theta_0 > 0\) the thermocapillarity weakens the tendency of formation of beads due to the Rayleigh-Plateau mechanism.

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1. Introduction

The problems of an exterior coating flow over the outside of a cylindrical fibre driven by gravity have recently received extensively scientific interest due to their importance to numerous industrial applications, for instance, in the processes of coating of insulation on a wire or a tube wall [1].

The coating flow of an isothermal fluid has been investigated experimentally and theoretically by many authors. Quere [2] has performed experiments to study the coating flow on a fibre. The results showed that the coating flow exhibits rich dynamics including the formation of droplets, or beads due to a surface tension driven mechanism (the Rayleigh-Plateau instability) [3].

The linear stability problem of an isothermal coating flow was investigated by Goren [4], Lin and Liu [5]. The results in Ref.[5] showed that the coating film is unstable due to capillary pinching when the wavenumber of the disturbance is smaller than the cut-off value.

Investigations of the nonlinear problem of the coating flow down the outside of the fibre have been performed by many authors in terms of the long-wave theory. In these investigations, three types of modeling equations have been developed: (i) thin film asymptotic models (Benney-like equation) [6–8], (ii) long-wave asymptotic models [9,10], (iii) integral models [11].

The works mentioned above on the dynamics of coating flows are limited to isothermal flows.

In many industrial processes, the coating flows may occur in a cooling environment. For example, in glass manufacturing process, the water sprays is commonly used to cool freshly drawn glass fibres [12]. In this case, the effect of thermocapillarity on the stability is of crucial importance to the coating process. Recently, the problem of the thermocapillary effect on the dynamics of coating flow over a heated fibre wall has given rise to broad scientific interest due to its technological importance [13–18].

Dívalos-Orozco and You [13] studied the three-dimensional instability of a coating flow over a heated vertical cylinder. It was interesting that in the absence of gravity the thermocapillarity is possible to promote a non-axisymmetric unstable mode. The nonlinear dynamics of the coating flow down a uniformly heated vertical fibre has been studied by Liu and Liu [14]. It was found that the thermocapillarity enforces the Rayleigh-Plateau instability.

Recently, Liu et al. [15] performed a spacial-temporal analysis to study the effect of thermocapillarity on the absolute-convective instabilities of the problem. The results showed that the thermocapillarity enforces the absolute instability and promotes the

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Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$a$</td>
<td>the radius of fibre, dimensional (N/m²) or dimensionless</td>
</tr>
<tr>
<td>$c$</td>
<td>the wave speed, dimensional (m/s) or dimensionless</td>
</tr>
<tr>
<td>$k$</td>
<td>the streamwise wavenumber, dimensional (m⁻¹) or dimensionless</td>
</tr>
<tr>
<td>$m$</td>
<td>the azimuthal wavenumber</td>
</tr>
<tr>
<td>$n, t$</td>
<td>unit vectors of normal and tangent to the surface</td>
</tr>
<tr>
<td>$p$</td>
<td>pressure of fluid, dimensional (N/m²) or dimensionless</td>
</tr>
<tr>
<td>$q$</td>
<td>Newton's heat transfer coefficient (W/m²K)</td>
</tr>
<tr>
<td>$r$</td>
<td>radial component of the polar coordinates, dimensional (m) or dimensionless</td>
</tr>
<tr>
<td>$u$</td>
<td>velocity vector, dimensional (m/s) or dimensionless</td>
</tr>
<tr>
<td>$u, v, w$</td>
<td>velocity components in $r, \theta, z$ directions, dimensional (m/s) or dimensionless</td>
</tr>
<tr>
<td>$z$</td>
<td>axial component of the polar coordinates, dimensional (m) or dimensionless</td>
</tr>
<tr>
<td>$H$</td>
<td>2$H$ is the principle curvature of the fluid surface, dimensional (m⁻¹) or dimensionless</td>
</tr>
<tr>
<td>$L$</td>
<td>the capillary length scale defined as $\sigma/\rho g R$, (m)</td>
</tr>
<tr>
<td>$R$</td>
<td>the radius of the initial fluid surface, dimensional (m) or dimensionless</td>
</tr>
<tr>
<td>$S$</td>
<td>the position of the fluid surface, dimensional (m) or dimensionless</td>
</tr>
<tr>
<td>$T$</td>
<td>the temperature, dimensional (K) or dimensionless</td>
</tr>
<tr>
<td>$T_a$</td>
<td>the temperature at the fibre wall, (K)</td>
</tr>
<tr>
<td>$T_i$</td>
<td>the temperature at the surface, (K)</td>
</tr>
<tr>
<td>$T_\infty$</td>
<td>the temperature of the ambient gas far from the surface, (K)</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>the stress tensor, dimensional (N/m²) or dimensionless</td>
</tr>
<tr>
<td>$V$</td>
<td>the characteristic scale related to the gravity-driven velocity, $\rho g R^2/\mu$ (m/s)</td>
</tr>
<tr>
<td>$Pr$</td>
<td>the Prandtl number defined as $v/\kappa$</td>
</tr>
<tr>
<td>$Re$</td>
<td>the Reynolds number defined as $\rho V L/\mu$</td>
</tr>
<tr>
<td>$Ma$</td>
<td>the Marangoni number defined as $2\gamma/\mu V$</td>
</tr>
<tr>
<td>$Bi$</td>
<td>the Biot number defined as $q R/\chi$</td>
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Greek symbols

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<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\gamma$</td>
<td>the parameter defined as $(1/2)\alpha^2 \sigma/\alpha T^2$</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>the parameter defined as $R/L$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>azimuthal angel of the polar coordinates</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>the thermal diffusivity (m²/s)</td>
</tr>
<tr>
<td>$\mu$</td>
<td>the dynamic viscosity (kg·m·s⁻¹)</td>
</tr>
<tr>
<td>$\nu$</td>
<td>the kinematic viscosity (m²/s)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>the density of the fluid (kg/m³)</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>the surface tension (N/m)</td>
</tr>
<tr>
<td>$\chi$</td>
<td>the thermal conductivity (W/(m·K))</td>
</tr>
<tr>
<td>$\omega$</td>
<td>the frequency, dimensional (s⁻¹) or dimensionless</td>
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Subscripts

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<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$0$</td>
<td>at the reference state</td>
</tr>
<tr>
<td>$i$</td>
<td>at the interface</td>
</tr>
<tr>
<td>$r, z, t$</td>
<td>derivatives on $r, z$ and $t$</td>
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Superscripts

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<th>Symbol</th>
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<tr>
<td>*</td>
<td>in dimensionless form</td>
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breakup of the film into smaller droplets. Ding and Wong [16] studied the three-dimensional dynamics of the coating flow over a vertical cylinder with Marangoni effect using a thin film model. The results of the nonlinear simulation revealed that symmetry-breaking phenomenon can occur in the nonlinear stage when the Marangoni number exceeds a critical value.

The theoretical works of thermocapillary instabilities in coating flows on cylinders mentioned above have been limited so far to fluids whose surface tension is approximated as a linearly decreasing function of temperature. We refer to this behaviour as ‘linear surface tension’. It is well known that there exist a vast number of liquids with nonlinear dependence of surface tension on temperature. For example, such anamolies of surface tension have been observed in nematic liquid crystals [19], water–oil–surfactant systems [20], ionic liquids [21] and dilute aqueous long-chain alcohols [22]. The surface tensions of fluids of dilute aqueous long-chain alcohols exhibit well-defined minima at specific temperatures [22]. The surface tension attains its minimum value, $\sigma_{\text{min}}$, at the temperature $T_0$, and the parameter

$$\gamma = (1/2)\alpha^2 \sigma/\alpha T^2$$

is a positive number. This behavior of quadratic surface tension is referred to as ‘self-rewetting’ by Abe [23] in the boiling processes. Zhang [24] suggested to use self-rewetting fluids as working fluids to improve the performance of heat-pipe systems and their operative stability. Oron and Rosenau [25] studied the nonlinear thermocapillary effect in thin liquid layers of self-rewetting fluids. It is interesting that for the films of self-rewetting fluids, perturbations of the film interface may evolve into continuous steady patterns that do not rupture. More recently, Batson et al. [26] investigated the dynamics of a thin film of self-rewetting fluid which is subjected to a temperature modulation in the bounding gas.

The coating process using fluids with ‘linear surface tension’ suffers from instability because the mutual reinforcement between the Rayleigh-Plateau mechanisms and the thermocapillarity induced by a negative surface-tension gradient with temperature [14]. When a self-rewetting fluid is chosen for the coating process, both the linear stability characteristics and nonlinear dynamics are altered. Our objective is to examine the dynamics of a promising candidate of using self-rewetting fluids as effective ways to improve the performance of coating and its operative stability.

The present paper is organized as follows. In Section 2, the mathematical formulation of the physical model is presented. In Section 3, we present the results and discussion. In Section 4, we summarize the results and present the conclusions.

2. Mathematical formulation

2.1. Governing equations

The sketch of the coating flow is shown in Fig. 1. The coating flow of a Newtonian fluid with constant viscosity $\mu$ and density $\rho$, is driven by gravity $g$ over a vertical fibre of radius $r = a$. The initial radius of the fluid surface measured from the fibre centreline is $R = r$. The temperatures at the fibre wall ($r = a$) and the surface of the film ($r = S$) are denoted by $T_a$ and $T_i$, respectively.

We assume that the flow is axisymmetric without any variation in the azimuthal $\theta$-direction and without the azimuthal velocity component. The dynamics of the flow are governed by the continuity equation, the Navier–Stokes equations and the energy equation:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\nabla\cdot \left( \rho \mathbf{u} \right) = 0,$$

$$\rho \frac{D \mathbf{u}}{D t} = -\nabla p + \nabla \cdot \left( \mu \nabla \mathbf{u} \right),$$

$$\rho C_p \frac{D T}{D t} = \nabla \cdot \left( \kappa \nabla T \right).$$
where $V_i$ is the surface gradient operator, the stress tensor $T = -\mu \mathbf{I} + \mu \left[ \mathbf{V}u + (\mathbf{V}u)^T \right]$, the unit vectors normal and tangent to the interface

$$n = \frac{(1,-S_t)}{(1 + S_t^2)^{1/2}},$$

$$t = \frac{(S_t, 1)}{(1 + S_t^2)^{1/2}}$$

and the surface principal curvature $2H$ is

$$2H = -\mathbf{V} \cdot n.$$  

For self-rewetting fluids, the surface tension is given by the quadratic dependence on the temperature as in Eq. (1).

The kinematic boundary condition at the free surface is

$$S_t + wS_r - u = 0.$$  

Eq. (12) can be written in conservative form as

$$S_t + \frac{1}{5} \frac{\partial}{\partial z} \int_a^b w r \, dr = 0.$$  

The heat balance at the free surface is given by Newton's law of cooling,

$$-n \cdot \mathbf{V}T = q(T - T_\infty),$$

in which $\gamma$ is the thermal conductivity of the liquid, $T_\infty$ is temperature of the ambient gas far away from the interface, and $q$ is the Newton's heat transfer coefficient.

### 2.2. The asymptotic long-wave equation

In the present paper, we follow Craster and Matar [10] and adopt a set of similar scalings

$$r = Rr^*, z = Lz^*, \rho = \rho g L^2, t = LV^{-1}t^*, w = Vw^*,$$

$$u = \epsilon V u^*, T - T_\infty = \Delta T_\Theta^*.$$  

In Eq. (15), the star denotes dimensionless variables. The characteristic velocity scale is related to the gravity-driven velocity, i.e., $V = \rho g L^2/\mu$. The radial length scale is chosen to be associated with the initial radius $R$ and the axial length scale $L$ is taken to be the capillary length $\sigma/\rho g R$. The temperature scale is taken to be the difference $T = T_a - T_\infty$. We assume the parameter $\epsilon = R/L$ to be small, which indicates the dominating effect of surface tension over gravity. It is notable that the choice of lengthscale above sets $Bo = \epsilon$. The assumption of small $\epsilon$ means that the choice of a low Bond number (surface-tension-dominated) is consistent with the long-wave theory. In the existing experiments by Kliakhandler et al. [9], for a film of silicone oil at 25°C with radius $R$ about 0.5–1 mm, the parameter $Bo$ is typically small ($\approx 0.3$ or so).

Substituting Eq. (15) into Eqs. (2)–(4), the governing equations for dimensionless variables are written as (hereafter we have dropped the star for simplicity)

$$u_t + \frac{u}{r} + w_z = 0,$$

$$\epsilon^4 Re (u_t + uw_t + wu_z) = -p_r + \epsilon^2 \left[ u_{rr} + \frac{u}{r^2} + \epsilon \frac{\partial^2 u}{\partial z^2} \right],$$

$$\epsilon^2 Re (w_i + uw_i + wu_x) = 1 - p_z + \left[ w_{rr} + \frac{w}{r} + \epsilon^2 w_z \right],$$

$$\epsilon^2 Re (\Theta_i + u\Theta_i + w\Theta_z) = \frac{1}{Pr} \left[ \Theta_{rr} + \frac{\Theta}{r} + \epsilon^2 \Theta_{zz} \right].$$

---

Fig. 1. Sketch of the geometry of a coating flow on a fibre.
where the Reynolds number \(Re\) and the Prandtl number \(Pr\) are defined as
\[
Re = \frac{\rho V L}{\mu},
\]
\[
Pr = \frac{\nu}{\kappa}.
\]

For surface-tension-dominated flows, \(\epsilon\) is a small number. In the existing experiments by Klikhander [9], the Reynolds number \(Re \sim 10^{-2}\). Assuming \(\epsilon^2 \ll 1\) and \(Re \sim O(1)\) or small, we can neglect the contribution of inertial terms and the streamwise viscous diffusion.

We obtain the leading-order equations and the boundary conditions as follows by dropping the terms of \(O(\epsilon^2)\) and higher
\[
w_t + \frac{w_r}{r} = p_r - 1, \tag{22}
\]
\[
\Theta_t + \frac{\Theta_r}{r} = 0. \tag{23}
\]

The boundary conditions \(r = a\) are
\[
w = 0, \tag{24}
\]
\[
\Theta = 1. \tag{25}
\]

The boundary conditions \(r = S\) are
\[
p = \frac{1}{5} - \epsilon^2 S_x, \tag{26}
\]
\[
w_r = Ma(\Theta_i - \Theta_0)(S_z \Theta_r + \Theta_{r2}). \tag{27}
\]
\[
\Theta_r + Bi \Theta = 0, \tag{28}
\]
where the Marangoni number which measures the importance of thermocapillarity is defined as \(Ma = \frac{\epsilon^2 a^2 r^2}{\nu^2}\) and the Biot number is defined as \(Bi = q R / \chi\). The Biot number is used to measure the efficiency of heat transfer at the boundary. The limiting cases of \(Bi = 0\) and \(Bi \rightarrow \infty\) corresponds to a poorly conducting case and a perfectly conducting case, respectively.

It notable that in Eq. (26) contribution of the streamwise curvature \(\epsilon^2 S_x\) due to surface tension must be kept even though it includes \(\epsilon^2\). It is well known that this term reflects the principal physical effect that prevents the waves from breaking in the planar case. Physically speaking, in certain regions the streamwise curvature is sufficiently large such as the steep front edge of a solitary hump, hence \(\epsilon^2 S_x\) cannot be neglected. The importance of this term in coating flows has been reflected in a linear analysis where the retaining of this term is vital to predict the correct cutoff wavenumber [10]. More justification of inclusion of this term has been reviewed by Craster and Matar [10] and Ruyer-Quil et al. [11].

The distribution of the temperature can be solved from Eq. (23) with boundary conditions (25) and (28),
\[
\Theta = \frac{Bi \ln \frac{a}{S}}{Bi \ln \frac{a}{S} + \frac{1}{3}}, \tag{29}
\]

The temperature at the interface, \(\Theta_i\), is
\[
\Theta_i = -\frac{1}{Bi \ln \frac{a}{S} + 1}. \tag{30}
\]

The velocity \(w(r,z,t)\) can be obtained by solving Eq. (22) with boundary conditions (24) and (27),
\[
w = (1 - p_r) \left[ \frac{1}{4} (a^2 - r^2) + \frac{1}{2} \epsilon^2 \ln \frac{r}{a} + Ma(\Theta_i - \Theta_0) \Theta_{r2} S \ln \frac{r}{a} \right]. \tag{31}
\]

Substituting \(w\) into the kinematic boundary condition (13) yields an evolution equation for \(S(z,t)\) given by
\[
\partial_t S^2 + 2\partial_z \left[ \frac{1}{4} S \ln \frac{r}{a} + \frac{35^2 - a^2}{16} (S^2 - a^2) \right] = 0. \tag{32}
\]

The flow is characterized by dimensionless parameters including the aspect ratio \(a\), Marangoni number Ma, the Biot number Bi and the Bo number Bo or \(\epsilon\).

3. Results and discussion

3.1. Linear stability

To gain a basic understanding on the effect of thermocapillarity on the dynamics of the flow, we begin with a brief examination on the linear stability of the problem. We use a normal mode approach:
\[
S = \bar{S} + \delta S \exp[i(kz - \omega t)], \tag{33}
\]
where \(\delta S\) is the amplitude of the perturbation, \(k\) and \(\omega\) are the wavenumber and the complex frequency, and the base state of the profile of interface is \(S = 1\).

Substituting Eq. (33) into Eq. (32) and dropping the nonlinear parts yields the dispersion relation
\[
-\omega + \left[ \frac{A}{16} k^2 \left( k^2 \epsilon^2 - 1 \right) + \frac{ik}{2} C \right] + \frac{1}{4} \epsilon^2 Ma(\Theta_i - \Theta_0) B = 0, \tag{34}
\]
in which
\[
A = \left( 4 \ln \frac{1}{a} - a^4 + 4a^2 - 3 \right), \tag{35}
\]
\[
B = \frac{Bi(a^2 - 1 - 2 \ln a)(\ln a - 1)}{Bi \ln a - a^2}, \tag{36}
\]
\[
C = a^2 - 1 - 2 \ln a, \tag{37}
\]
are positive coefficients.

We use \(Re(\omega)\) and \(Im(\omega)\) to denote the real and imaginary parts of \(\omega\), respectively. \(Re(\omega)\) corresponds to the frequency, and \(Im(\omega)\) corresponds to the time-growth rate. The growth rate
\[
Im(\omega) = \frac{\epsilon^2 A}{16} k^2 \left( \epsilon^2 F - k^2 \right), \tag{38}
\]
in which
\[
F = 1 - 4Ma(\Theta_i - \Theta_0) \frac{B}{A}. \tag{39}
\]

At \(Ma = 0\), this dispersion relation is identical to that of Craster and Matar [10] for the isothermal case.

As \(F \ll 0\), the maximum growth rate of \(Im(\omega) = 0\) occurs at \(k = 0\). The growth rate \(Im(\omega) \leq 0\) for all \(k\). In this case, the film flow is linearly stable. As \(F > 0\), the film flow is linearly unstable. The cut-off wavenumber for which the real growth rate is zero is
\[
k_c = 1.6 F^{1/2}. \tag{40}
\]

The maximum growth rate
\[
max(Im(\omega)) = \frac{A}{64\epsilon^2} F^2, \tag{41}
\]
is realized at the wavenumber
\[
k_m = \frac{1}{\sqrt{2\epsilon}} F^{1/2}. \tag{42}
\]
two typical cases of interfacial flow at the interface of the film of a self-rewetting fluid for stress on adjacent fluids. As shown in Fig. 3(a) for a given Ma, the growth rate increases or decreases with the increase of Ma for $\Theta_i - \Theta_0 < 0$ and $\Theta_i - \Theta_0 > 0$. In Fig. 3(a) for $\Theta_i - \Theta_0 < 0$, it is obvious that the growth rate increases with the increase of Ma. This means that thermo-capillarity is destabilizing when $\Theta_i - \Theta_0 < 0$. In Fig. 3(b) for $\Theta_i - \Theta_0 > 0$, the effect of thermo-capillarity is stabilizing as it is observed that the growth rate decreases with the increase of Ma.

In order to understand the results of the linear stability analysis, it is helpful to give a physical interpretation on the mechanisms of the Marangoni effect on the stability. We begin by analyzing the dynamics of a small disturbance at the interface of the coating flow. In Fig. 3(a) and (b), we present the schematic of thermocapillary flow at the interface of the film of a self-rewetting fluid for two typical cases of $\Theta_i - \Theta_0 < 0$ and $\Theta_i - \Theta_0 > 0$. At the interface, crest and trough are formed due to the growth of the deflexion driven by the Rayleigh-Plateau mechanism. When the film is heated by the fibre, because the trough point is closer to the fibre wall, the temperature at the trough is higher than that at the crest. A fluctuation of temperature will result in a local surface tension gradient. Surface tension gradients at the interface act as tangential stress on adjacent fluids. As shown in Fig. 3(a) for $\Theta_i - \Theta_0 < 0$, the interface relaxes at the trough point where the surface tension decreases and then the thermocapillarity induces the interfacial flow from the trough towards the crest. The deflexion driven by the Rayleigh-Plateau mechanism is amplified by this type of interfacial flow. Thus, we conjecture that the Marangoni effect and the Rayleigh-Plateau mechanism reinforce the instability of flow for $\Theta_i - \Theta_0 < 0$.

However, in Fig. 3(b) for $\Theta_i - \Theta_0 > 0$, the maximum surface tension occurs at the trough and the interface relaxes at the crest point. In this case, the deflexion driven by the Rayleigh-Plateau mechanism is weakened by this type of interfacial flow. It can be expected that the Marangoni effect weakens the Rayleigh-Plateau instability for $\Theta_i - \Theta_0 > 0$.

In Fig. 4, we plot the marginal curves between the stable and unstable regions in the $k_c - Ma$ plane. For $\Theta_i - \Theta_0 < 0$, the cut-off wavenumber $k_c$ increases with the increase of Ma. For $\Theta_i - \Theta_0 = 0$, the cut-off wavenumber is independent of Ma. For $\Theta_i - \Theta_0 > 0$, the cut-off wavenumber $k_c$ decreases with the increase of Ma. As the Marangoni number exceeds the critical value $Ma_c$ defined as

$$ Ma_c = \frac{A}{4B} \frac{1}{\Theta_i - \Theta_0} $$

the Rayleigh-Plateau instability is completely suppressed by the thermocapillarity.

### 3.2. Nonlinear evolution of the most unstable mode

In this subsection, we performed numerical simulations to study the effect of the thermocapillarity on the characteristics of the nonlinear dynamics of the flow. We imposed simple periodic conditions in the simulations with the computational domain set to be the interval $[0, \pi]$. For periodic problems, the position of interface can be approximated by Fourier series:

$$ S(z, t) = \sum_{n=0}^{N-1} S_n(t) \exp(i2\pi nz/\ell). $$

We define the wave speed $c = Re(\omega)/k$. From Eq. (34), the wave speed

$$ c = \frac{a^2 - 1 - 2\ln a}{2}. $$

This means that the wave speed is only dependent on the geometric parameter $a$.

From Eq. (38) and (39) that for a given $k$, the growth rate increases or decreases with the increase of Ma for $\Theta_i - \Theta_0$ is negative or positive. In Fig. 2(a) and (b), we show the curves of the dispersion relations for two typical cases of $\Theta_i - \Theta_0 < 0$ and $\Theta_i - \Theta_0 > 0$. In Fig. 2(a) for $\Theta_i - \Theta_0 < 0$, it is obvious that the growth rate increases with the increase of Ma. This means that thermo-capillarity is destabilizing when $\Theta_i - \Theta_0 < 0$. In Fig. 2(b) for $\Theta_i - \Theta_0 > 0$, the effect of thermo-capillarity is stabilizing as it is observed that the growth rate decreases with the increase of Ma.

In order to understand the results of the linear stability analysis, it is helpful to give a physical interpretation on the mechanisms of the Marangoni effect on the stability. We begin by analyzing the dynamics of a small disturbance at the interface of the coating flow. In Fig. 3(a) and (b), we present the schematic of thermocapillary flow at the interface of the film of a self-rewetting fluid for two typical cases of $\Theta_i - \Theta_0 < 0$ and $\Theta_i - \Theta_0 > 0$. At the interface, crest and trough are formed due to the growth of the deflexion driven by the Rayleigh-Plateau mechanism. When the film is heated by the fibre, because the trough point is closer to the fibre wall, the temperature at the trough is higher than that at the crest. A fluctuation of temperature will result in a local surface tension gradient. Surface tension gradients at the interface act as tangential stress on adjacent fluids. As shown in Fig. 3(a) for $\Theta_i - \Theta_0 < 0$, the interface relaxes at the trough point where the surface tension decreases and then the thermocapillarity induces the interfacial flow from the trough towards the crest. The deflexion driven by the Rayleigh-Plateau mechanism is amplified by this type of interfacial flow. Thus, we conjecture that the Marangoni effect and the Rayleigh-Plateau mechanism reinforce the instability of flow for $\Theta_i - \Theta_0 < 0$.

However, in Fig. 3(b) for $\Theta_i - \Theta_0 > 0$, the maximum surface tension occurs at the trough and the interface relaxes at the crest point. In this case, the deflexion driven by the Rayleigh-Plateau mechanism is weakened by this type of interfacial flow. It can be expected that the Marangoni effect weakens the Rayleigh-Plateau instability for $\Theta_i - \Theta_0 > 0$.

In Fig. 4, we plot the marginal curves between the stable and unstable regions in the $k_c - Ma$ plane. For $\Theta_i - \Theta_0 < 0$, the cut-off wavenumber $k_c$ increases with the increase of Ma. For $\Theta_i - \Theta_0 = 0$, the cut-off wavenumber is independent of Ma. For $\Theta_i - \Theta_0 > 0$, the cut-off wavenumber $k_c$ decreases with the increase of Ma. As the Marangoni number exceeds the critical value $Ma_c$ defined as

$$ Ma_c = \frac{A}{4B} \frac{1}{\Theta_i - \Theta_0} $$

the Rayleigh-Plateau instability is completely suppressed by the thermocapillarity.

### 3.2. Nonlinear evolution of the most unstable mode

In this subsection, we performed numerical simulations to study the effect of the thermocapillarity on the characteristics of the nonlinear dynamics of the flow. We imposed simple periodic conditions in the simulations with the computational domain set to be the interval $[0, \pi]$. For periodic problems, the position of interface can be approximated by Fourier series:

$$ S(z, t) = \sum_{n=0}^{N-1} S_n(t) \exp(i2\pi nz/\ell). $$

We define the wave speed $c = Re(\omega)/k$. From Eq. (34), the wave speed

$$ c = \frac{a^2 - 1 - 2\ln a}{2}. $$

This means that the wave speed is only dependent on the geometric parameter $a$.
in which $\hat{s}_n$ is the Fourier coefficient and $N$ is the number of Fourier modes. A Fourier pseudospectral method is used for the discretization in space. The nonlinear terms are efficiently treated by FFT. The second-order Runge–Kutta method for stiff problems (Gear method) was used for the time stepping with the relative error set to be less than 10\(^{-6}\).

The results of linear stability analysis showed that the thermocapillarity does not influence the wave speed and only changes the time growth rate of the small disturbance. It is interesting to know how the thermocapillarity influences the dynamics of the coating flow of self-rewetting fluids. We now examine the nonlinear evolution initiated by small disturbances of the most unstable modes with wavelength $\lambda = 2\pi/k_0$. The initial condition is a simple harmonic disturbance superimposed on the interface

$$S_0 = \tilde{S} + \delta \cos(k_m z),$$

(45)

where $\delta$ is a small number.

In Fig. 5, we plot the profiles of the saturated state of the nonlinear evolution initiated by small disturbances of the most unstable modes. At the saturated state, the evolution of the interface is in the form of a traveling wave. In Fig. 5(a) for $\Theta_1 - \Theta_0 < 0$, with the increase of $Ma$ the height of the beads increases and the gap between drops becomes flatter. This means that the thermocapillarity promotes the formation of the beads. In Fig. 5(b) for $\Theta_1 - \Theta_0 > 0$, it is observed that with the increase of $Ma$ the deformation of the interface becomes weaker.

In Fig. 6, we plot the profiles of the interface and streamlines for several typical cases. In Fig. 6(a) for $\Theta_1 - \Theta_0 < 0$ and $Ma = 0$, a small capillary ripple exists in the region wherein the droplet adjusts onto the preceding flat region. There is a small region wherein the flow is stagnant near the capillary ripple. As $Ma$ increases to 1.0, in Fig. 6(b) the height of the bead increases and the thickness of gap region decreases from about 0.3 to 0.1. Comparing Fig. 6(b) with (a), it is found that for $Ma = 1.0$ the gap region becomes flatter and the beads-like structure is more pronounced. In Fig. 6(c) for $\Theta_1 - \Theta_0 > 0$ and $Ma = 1.0$, the height of the bead is less than that in Fig. 6(a). As $Ma$ increases to 3.0 in Fig. 6(d), the interface is in a form of a wavy surface and the beads-like structure is less pronounced. As $Ma$ increases further, the interface becomes flatter until the disturbance are suppressed by the thermocapillarity. Fig. 6(c) and (d) show that for $\Theta_1 - \Theta_0 > 0$, the effect of thermocapillarity can weaken the tendency of formation of beads. This result is consistent with that of the linear stability analysis in which the thermocapillarity is stabilizing for self-rewetting fluid when $\Theta_1 - \Theta_0 > 0$.

In Fig. 7, we plot the wave speed $c$ of the nonlinear saturated state obtained from numerical simulations for various values of $Ma$. The wave speed predicted by linear stability analysis is given by the dashed line in Fig. 7. At $Ma = 0$, the values of wave speed of the nonlinear saturated state are obviously lower than that predicted by the linear stability analysis. For $\Theta_1 - \Theta_0 < 0$ as shown by curve ‘a’ the wave speed $c$ gradually decreases with the increase of $Ma$. For $\Theta_1 - \Theta_0 > 0$, with the increase of $Ma$ the wave speed increases to a maximum value at $Ma = 2.7$. As $Ma$ increases further the wave speed gradually decreases and the curve approaches to the linear wave speed.

In order to know the characteristics of the nonlinear evolution of disturbance with different wavelength $l$, we performed numerical simulations for evolutions with different wavelengths. The linear stability analysis showed that there exists a minimum wavelength $l_c = 2\pi/k_c$ below which the flow is stable. It is found that the interface is stable if the wavelength $l$ is shorter than $l_c$. This result is consistent with the result predicted by the linear stability analysis.

In Fig. 8, the wave speed versus the wavelength in the range of $l_c < l < 2l_c$ are plotted for different $Ma$. In this range of wavelength, the solution is in the form of a single-hump travelling wave which is insensitive to initial conditions. As shown in Fig. 8(a) for $\Theta_1 - \Theta_0 < 0$, for all $Ma$ the wave speed increases with the decrease of $k/k_c$ and at a given $k/k_c$ the wave speed decreases with the increase of $Ma$. In Fig. 8(b) for $\Theta_1 - \Theta_0 > 0$, the behaviours of the wave speed with the various of $Ma$ are complicate. In the vicinity of $k/k_c = 0.5$, for $Ma = 1.0$ and 2.0 the wave speed decreases and
reach its minimum value, and then gradually increases with the increase of $k/k_c$. As $Ma$ increases to 3.0, the value of $c$ monotonously decreases with the increase of $k/k_c$. As $k/k_c$ approaches to 1.0, the curves with different $Ma$ approach to the value of the wave speed predicted by the linear stability analysis. As $Ma$ increases further to 3.98, the instability is completely suppressed by the thermocapillarity and the wave speed becomes a constant value which corresponds to the wave speed predicted by the linear stability analysis.

3.3. Coherent structure: travelling wave solutions

Steadily propagating droplet or bead-like solutions separated by long gaps of constant radius have been observed in previous experiments by Kliakhandler et al. [9]. In many types of film flows, there exist various families of travelling solutions with different wavelengths and wave speeds. In this subsection, we will examine the effect of thermocapillarity on the characteristics of the travelling wave solutions. In the case of regularly propagating wave with constant velocity $c$, solution to the nonlinear Eq. (32) is sought in form of $S(\zeta)$ in which $\zeta = z - ct$. We will solve the evolution equation by moving to a travelling wave coordinate, $\zeta = z - ct$, where $c$ is to be determined. The nonlinear Eq. (32) becomes

$$-c\partial_\zeta S^2 + 2\partial_\zeta Q(S) = 0,$$

in which

$$Q(S) = (1 - p_c) \left[ \frac{1}{4} S^4 \ln \frac{S}{a} + \frac{3S^2 - a^2}{2} \right] \left[ \frac{a^2 - S^2}{6} \right] + Ma(\Theta_i - \Theta_b) \partial_\zeta \left[ \frac{S^4}{2} \ln \frac{S}{a} - \frac{S^2}{4} \left( S^2 - a^2 \right) \right].$$

Integrating Eq. (44), we have

$$-cS^2 + 2Q(S) = q.$$  (48)

Eq. (48) becomes a nonlinear eigenvalue problem where the unknown variables are $S(\zeta), q$ and $c$. In order to fix $c$, we have to impose an additional constraint condition on the fluid mass,

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} \left( S^2(\zeta) - a^2 \right) d\zeta = l(1 - a^2).$$  (49)

This nonlinear eigenvalue problem subject to periodic boundary conditions will be solved in the region of $-1/2 < \zeta < 1/2$, where $l$ is the length of the computational domain.

A Newton-Kantorovich approach with a Fourier expansion is efficient to solve the nonlinear eigenvalue problem with periodic boundary conditions. The travelling wave solution can be rapidly converged via Newton iterations if the initial guess is reasonable. We used the wave speed $c$ and profile $S(\zeta)$ obtained from numerical simulation as a initial guess for the iteration procedure.

In Fig. 9, we plot the curves of travelling wave solutions in the $c - l$ parametric plane for various Marangoni numbers. In Fig. 9 (a) for $\Theta_i - \Theta_b < 0$, the structures of the curves with smaller values of $Ma = 0, 0.5$ and 1.0 are similar. It is obvious that solutions with larger $l$ travel faster than those with smaller $l$ for $Ma = 0, 0.5$ and 1.0. This means that relatively long travelling waves eventually catch up with shorter ones. The curves of $Ma = 0, 0.5$ and 1.0 also show that for long-wave solutions the wave speed increases with the increase of $Ma$. However, as $Ma$ increase to 2.0 or more, the

Fig. 6. The profiles of the interface and the streamlines. (a) $Ma = 0, \Theta_i - \Theta_b = -0.5$, (b) $Ma = 1, \Theta_i - \Theta_b = -0.5$, (c) $Ma = 1, \Theta_i - \Theta_b = 0.2$, (d) $Ma = 3, \Theta_i - \Theta_b = 0.2$. The other parameters are $Bo = 0.2, a = 0.3, Bi = 1.0$.

Fig. 7. The curves of the wave speed $c$ for various $Ma$. The curves labeled by squares and circles are given by the wave speeds of nonlinear evolution of the most unstable modes. The curve of the wave speed predicted by the linear dispersion relation is plotted by the dashed line. (a) $\Theta_i - \Theta_b = -0.5$, (b) $\Theta_i - \Theta_b = 0.2$. The other parameters are $Bo = 0.2, a = 0.3$. 

curves become wavy in the $c - l$ plane and the travelling wave solutions only exist in a limited band of wavelength.

In Fig. 9(b) for $\Theta_i - \Theta_o < 0$, the structures of the curves with various $Ma$ are similar. With the increase of $Ma$, the curves become more steepen in the $c - l$ plane. In the long-wave range the travelling wave solution propagates slower with the increase of $Ma$. As $Ma$ exceeds the critical value, the flow becomes stable and the curve of travelling wave solution disappears in the $c - l$ plane.

4. Conclusions

In the present paper, the problem of the dynamics of an exterior coating flow of a self-rewetting fluid over a fibre has been investigated. It is assumed that the characteristic radius of fluid ring is much smaller than the capillary lengthscale. Under this assumption, the evolution equation of the position of the free surface is derived in the framework of the long-wave theory.

A linear stability analysis is performed to examine the influence of thermocapillarity on the stabilities of small axisymmetric disturbances. It is found that the thermocapillarity plays an important role in influencing the time growth rate without changing the wave speed of the disturbance. For $\Theta_i - \Theta_o < 0$, the time growth rate increases with the increase of $Ma$ and the thermocapillarity is destabilizing. For $\Theta_i - \Theta_o > 0$, the thermocapillarity is stabilizing. As $Ma$ exceeds a critical value, the Rayleigh-Plateau instability can be suppressed by the thermocapillarity.

We also performed numerical simulations on the nonlinear evolution to examine the influence of thermocapillarity on the characteristics of the nonlinear dynamics of the problem. At the saturated state, the interface is in the form of a series of travelling waves. For $\Theta_i - \Theta_o < 0$, the thermocapillarity promotes more pronounced beads-like structures. However, For $\Theta_i - \Theta_o > 0$, the thermocapillarity weakens the tendency of formation of beads. A series of travelling waves with different wave length were obtained via solving nonlinear eigenvalue problems in which the eigenvalue corresponds to the wave speed. The effect of thermocapillarity on the characteristics of the travelling wave solutions for $\Theta_i - \Theta_o < 0$ is different from that for $\Theta_i - \Theta_o > 0$. For $\Theta_i - \Theta_o < 0$, with the increase of $Ma$ the curves of the travelling wave solutions become wavy in the $c - l$ plane and exist in a limited band of wavelength. As $Ma$ exceeds the critical value, the curve of the travelling wave solution disappears as the film becomes stable.

The results of the linear stability analysis and the nonlinear simulations indicate that when the operating temperature at the interface is higher than $T_0$, using self-rewetting fluids is favorable for the coating process because the thermocapillarity is capable to offset or even suppress the Rayleigh-Plateau instability.

In our further work, experiments of the coating flow of a self-rewetting fluid are needed to be performed, because the experimental conditions are more complicated than the theoretical models.

Conflict of Interest

The authors declared that there is no conflict of interest.
Appendix A. Stability of the linearized Navier-Stokes equations

In the foregoing part of the present paper, we limit the results to axisymmetric disturbances. In this appendix, we perform a linear stability analysis on the linearized Navier-Stokes equations for both axisymmetric and non-axisymmetric disturbances. The velocity field \( u, v, w, p, \Theta \) and the position of the interface \( S \) are perturbed by infinitesimal harmonic disturbances as

\[
[u, v, w, p, \Theta, S] = \left[ \bar{u}, \bar{v}, \bar{w}, \bar{p}, \bar{\Theta}, \bar{S} \right] + \left[ \tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}, \tilde{\Theta}, \tilde{S} \right]
\]

where the bar, “\( \bar{\} \)”, refers to the base state and the hat, “\( \hat{\} \)”, denotes the Fourier amplitudes of the disturbances, \( \omega \) denotes the frequency, and the integer \( m \) and the real number \( k \) are the azimuthal and streamwise wavenumbers, respectively. The basic states of \( w \) and \( \Theta \) are

\[
\bar{w} = \frac{1}{4} \left( \omega^2 - r^2 \right) + \frac{1}{2} \ln \frac{r}{a}.
\]

\[
\bar{\Theta} = \frac{Bl \ln r - 1}{Bl \ln a - 1}.
\]

The fully linearized governing equations for the disturbances are as follows

\[
\frac{\partial \tilde{u}}{\partial t} + \frac{i m}{r} \tilde{v} + i k \tilde{w} = 0.
\]

\[
e^2 \text{Re}(-i \omega \tilde{u} + i k \tilde{w}) = -i \tilde{p}
\]

\[
e^2 \text{Re}(\tilde{u}) + i k \text{Im}(\tilde{w}) = -i \tilde{p}
\]

\[
\frac{\partial \tilde{v}}{\partial t} + \frac{\text{Im}(\tilde{w})}{r} = \left( e^{2k^2 + m^2 + 1} \right) \tilde{v} - \frac{2 i m}{r^2} \tilde{u}.
\]

\[
e^2 \text{Re}(-i \omega \tilde{v} + i k \tilde{w}) = -i \tilde{p}
\]

\[
e^2 \text{Re}(\tilde{v}) + i k \text{Im}(\tilde{w}) = -i \tilde{p}
\]

\[
\frac{\partial \tilde{w}}{\partial t} + \frac{\text{Im}(\tilde{w})}{r} = \left( e^{2k^2 + m^2 + 1} \right) \tilde{w}.
\]

\[
e^2 \text{Re}(-i \omega \tilde{w} + i k \tilde{w}) = -i \tilde{p}
\]

\[
\text{Re}(-i \omega \tilde{w} + i k \tilde{w}) = -i \tilde{p}
\]

\[
\frac{\partial \tilde{p}}{\partial t} + 2 \text{Re} \left( \frac{\partial \tilde{u}}{\partial t} + \frac{i m}{r} \tilde{v} + i k \tilde{w} \right) = \left( -1 + m^2 + e^{2k^2} \right) \tilde{S}.
\]

\[
\frac{\partial \tilde{w}}{\partial t} + \frac{i m}{r} \tilde{w} = i k Ma (\Theta_1 - \Theta_2) \left( \tilde{\Theta} + i \Theta \tilde{S} \right).
\]

\[
\frac{\partial \tilde{S}}{\partial t} + \frac{i m}{r} \tilde{S} = i k Ma (\Theta_1 - \Theta_2) \left( \tilde{\Theta} + i \Theta \tilde{S} \right).
\]
\[ \left( \mathcal{D} \Theta + B \Theta \right) + \left( \mathcal{I}^2 \Theta + B^2 \Theta \right) \delta = 0, \quad (A.13) \]

\[ -i \sigma \delta + ikw \delta - \dot{u} = 0. \quad (A.14) \]

The fully linearized equations are solved by a Chebyshev collocation method. In the long wave Eq. (32), it is assumed that \( c^2 \ll 1 \). In most of our computation, the parameter \( Bo \) or \( e \) is set to be 0.2. We will test the validity of the long wave model by comparing the results of dispersion relation predicted by Eq. (32) and the linear stability characteristics of the Navier-Stokes equations. We compared the dispersion relations for both models for \( Bo = 0.1, 0.2 \) and 0.3 as shown in Fig. A.10. Inspection of the dispersion curves shows that the curves of both models almost coincide as \( Bo \ll 0.3 \).

In the present paper, we assumed that the flow is axisymmetric and limited our results to the axisymmetrical cases. In this appendix, we will examine the stability of the flow for both axisymmetrical and non-axisymmetrical disturbances. We have computed the dispersion relations with different azimuthal modes from \( m = 0 \) to \( n \). In Fig. A.11, we plot the curves of the dispersion relations of the first two leading modes with \( m = 0 \) and 1. It is found that the most unstable mode with \( m = 0 \) is the detrimental mode. This result is consistent with the observations in isothermal coating flows in which the interfaces are always axisymmetric.

### Appendix B. Supplementary material

Supplementary data associated with this article can be found, in the online version, at https://doi.org/10.1016/j.ijheatmasstransfer.2019.03.049.

### References