Theoretical Investigation of Leak’s Impact on Normal Modes of a Water–Filled Pipe: Small to Large Leak Impedance

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Abstract: Recent research shows the potential of resonant frequency-based leakage detection methods. However, there is a disagreement in whether a leak shifts the normal modes (often called natural or resonant modes) and whether a leak introduces additional peaks to the frequency response function (FRF) of the pipeline. In this paper, the impact of a leak on the normal modes is investigated. The trajectories of normal modes in the frequency complex plane with varying leak size are studied. The key parameter that represents the leak size and controls the trajectories of the normal modes is the ratio of the acoustic impedance of the pipe to the resistance impedance of the leak. It is found that, as the impedance ratio increases from zero (i.e., no leak), each normal mode shifts toward the upper-half complex plane of frequency by a leak, where the imaginary part is a measure of the leak-induced damping of the wave. When the impedance ratio is less than the order of one, the leak-induced normal-mode frequency shift is negligible, which supports the theory put forward by proponents of the no-shift and no-additional-peak hypothesis. When the impedance ratio is of the order of one or larger, not only is the shift of the FRF’s peak significant, but also new peaks appear, which supports the theory raised by proponents of the leak-induced additional peaks hypothesis.

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Introduction

The problem of identifying leaks in water supply pipelines has been an active area of both research and technology development since the 1990s. The frequency domain leakage detection method has increasingly become a topic of research and has application potential in a wide range of systems (Chaudhry 1987; Jönsson and Larson 1992; Mpesha et al. 2001; Ferrante and Brunone 2003; Covas et al. 2005; Lee et al. 2005a, b; Sattar and Chaudhry 2010; Duan et al. 2011; Rubio Scota et al. 2016; Wang and Ghidaoui 2018a, b; Wang et al. 2019). However, debate remains on whether a leak in a pipeline gives rise to normal-mode frequency (resonant frequency) shift and additional peaks in the frequency response function (FRF).

In Lee et al. (2005b), a leak in the pipeline was assumed to be so small that it does not change the locations of peaks of the FRF. The numerical examples in Lee et al. (2005b) considered certain values of wave speed, leak size, and driving head; that is, the wave speed was 1,000 m/s, head at the leak was greater than 40 m, and the ratio between the effective leak size and pipe area was less than $5 \times 10^{-3}$. The choice of these values made the leak-related terms in the FRF comparatively small and thus negligible. As a result, Lee et al. (2005b) modified the FRF of a leaking pipeline system based on the assumption of no resonant frequency shift and no additional resonant frequency, by which a leak detection method was proposed.

On the contrary, Mpesha et al. (2001, 2002) proposed an opposite view that in the presence of a leak, the FRF is significantly changed and additional resonant peaks are found. Mpesha et al. (2001) developed a leakage detection method, where the leak locations and sizes were estimated using the information provided by the observed additional resonant peaks. Doubts regarding the validity of the method proposed by Mpesha et al. (2002) were raised in a discussion in Lee et al. (2010). Lee et al. (2010) claimed that studies such as Ferrante et al. (2001) and Lee et al. (2002) did not show additional resonant peaks due to the presence of a leak. These studies are in conflict with the finding in Mpesha et al. (2002) showing that a leak within pipeline systems gives rise to additional resonant peaks in the frequency response diagram. In the closure, M. H. Chaudhry argued that wave reflection and transmission from any geometrical change (such as a leak) in the pipe system result in a change in its frequency response. In the presence of a leak in a piping system, a wave gets reflection and transmission at this leak-formed additional boundary and finally induces changes in the piping system’s frequency response. Zecchin et al. (2008) showed that a leak affects only the eigenfrequencies of a system and that no new eigenfrequencies are introduced as a result of the leak.

In a related work, Ferrante and Brunone (2003) presented experimental results on the leak-induced impact on the frequency response of a pipeline system. They showed that for a very large leak, for example, the flow from the leak is 80% of the pipe base flow, the FRF significantly changes, and new peaks appear. This conclusion is further supported by the limiting case of leak size shown in Zecchin et al. (2008).

The current paper investigates the leak’s impact on the normal modes of a pipeline system and resolves the conflict that exists in the literature in relation to the impact of a leak on the wave...
in a pressurized pipe. In the physics literature, normal-mode frequencies are also referred to as natural or resonant frequencies (e.g., French 1971). This paper follows this tradition.

Normal Mode for Intact and Leaking Pipeline (Homogeneous Solution)

Consider the pipeline illustrated in Fig. 1, where friction is neglected in order to focus the analysis solely on the effect of leak. The frequency-domain solution to the water hammer equation is (Wylie and Streeter 1978)

\[
\begin{align*}
q^D &= \left[ \begin{array}{c}
\cosh(\gamma(l-x_L)) \\
\sinh(\gamma(l-x_L))
\end{array} \right], \\
\frac{h^D}{h^0} &= \left[ \begin{array}{c}
\frac{-1}{Z_C} \sinh(\gamma(l-x_L)) \\
\sinh(\gamma(l-x_L))
\end{array} \right]
\end{align*}
\]

where \( q \) and \( h \) = complex discharge and head perturbation in the wavenumber domain; the superscripts \( U \) and \( D \) stand for the upstream and downstream nodes; \( l \) = pipe length; and \( x_L \) = location of the leak away from the upstream reservoir. In addition, \( \gamma \) = propagation operator and is related to the complex-value temporal angular frequency \( \omega \) by \( \gamma = \omega/a \), where \( a \) is the group velocity of the wave, also known as wave speed (Tijsseling et al. 2010). The real and imaginary parts of \( \gamma \) represent the damping and phase change, respectively (Wylie and Streeter 1978; Chaudhry 2014). The value \( Z_C = a/(gA) \) = impedance of the pipe; \( A \) = cross-sectional area of the pipe; \( g \) = gravitational acceleration; and \( Z_L = H_L/Q_L \) is the impedance of the leak, where \( H_L \) and \( Q_L \) are the steady-state head and discharge at the leak, respectively. Both \( Z_C \) and \( Z_L \) are real-valued parameters.

Let \( k = \gamma/l = -i\gamma \).

As a result

\[
\cosh(\gamma) = \cosh(ik) = \frac{e^{ik} + e^{-ik}}{2} = \frac{1}{2} (\cos k + i \sin k) + \frac{1}{2} (\cos k - i \sin k) = \cos k
\]

and

\[
\sinh(\gamma) = \sinh(ik) = \frac{e^{ik} - e^{-ik}}{2} = \frac{1}{2} (\cos k + i \sin k) - \frac{1}{2} (\cos k - i \sin k) = i \sin k
\]

Therefore, Eq. (1) can be written as follows:

\[
\begin{align*}
q^D &= \left[ \begin{array}{c}
\cos(k(l-x_L)) - iZ_C \sin(k(l-x_L)) \\
-iZ_C \sin(k(l-x_L)) \cos(k(l-x_L))
\end{array} \right] \\
\frac{h^D}{h^0} &= \left[ \begin{array}{c}
\frac{-1}{Z_C} \sinh(k(l-x_L)) \\
\sinh(k(l-x_L))
\end{array} \right]
\end{align*}
\]

By enforcing the fixed-head boundary condition at the upstream (reservoir) \( h^U = 0 \), we obtain

\[
q^D = \left[ \begin{array}{c}
\cos(kl) + i\frac{Z_C}{Z_L} \cos(k(l-x_L)) \sin(kx_L) \\
Z_C \sin(k(l-x_L)) \cos(kx_L)
\end{array} \right] q^U
\]

and

\[
\begin{align*}
\frac{h^D}{h^0} &= \left[ \begin{array}{c}
-iZ_C \sin(kl) + \frac{Z_C^2}{Z_L^2} \sin(k(l-x_L)) \sin(kx_L)
\end{array} \right] q^U
\end{align*}
\]

For a closed valve at the downstream end, \( q^D = 0 \), and because \( q^U \neq 0 \), Eq. (5) leads to

\[
f(k) \triangleq \cos(kl) + i\frac{Z_C}{Z_L} \cos(k(l-x_L)) \sin(kx_L) = 0
\]

Eq. (7) is the nonforcing condition, which is also referred to as the homogeneous condition. Normal-mode wavenumbers are those wavenumbers \( k \) satisfying Eq. (7), which are in general complex numbers.

Case with No Leak

For the no-leak case, that is, \( Q_L = 0 \) (i.e., \( Z_C/Z_L = 0 \)), the second term in Eq. (7) becomes 0. As a result, the normal modes satisfy

\[
f(k) = \cos(kl) = 0
\]

which gives the normal-mode wavenumbers.
Here, Re and Im denote the real and imaginary part, and the superscript NL means no leak. This is a well-known result and can be found in the classic books (Chaudhry 1987; Wylie and Streeter 1978) and in many papers. The statement $\text{Im}(k_{NL}^n) = 0$ implies that there is no damping in the system.

**Case with Leak**

In the presence of a leak, $Z_C / Z_L \neq 0$, the roots of $f(k) = 0$ are complex numbers. The normal-mode wavenumbers are obtained by solving for the zeros of the Eq. (7), that is, $f(k) = 0$. In this paper, a logarithmic residue based quadrature method (LRBQM) (Kravanja and Van Barel 2007) is used to solve for the roots of $f(k) = 0$, $k \in C^+$. The LRBQM is based on Cauchy's residue theorem; a brief description of this method can be found in Appendix I. Using the LRBQM, a root of $f(k) = 0$ is computed via a numerical integration

$$k_n^L = \frac{1}{2\pi i} \oint_{C_n} \frac{k f'(k)}{f(k)} \, dk, \quad n \in \mathbb{N}^+$$

(10)

where $\delta C_n$ represents the boundary of $C_n$; and $C_n = \text{neighborhood of } k_{NL}^n$ including one and only one singularity of the integrand of Eq. (10). The number of singularities can be decided via Eq. (31).

Then, the leak induced shift of complex wavenumber can be calculated by

$$\Delta k_n^L = k_n^L - k_n^L = \frac{1}{2\pi i} \oint_{C_n} \frac{k f'(k)}{f(k)} \, dk - k_{NL}^n, \quad n \in \mathbb{N}^+$$

(11)

By computing the roots of $f(k) = 0$ for a leaking pipe, it is found that the normal-mode wavenumbers of pipe shift from the real line (in the case with no leak) toward the upper complex plane (in the case with a leak), which is detailed in the following example.

Consider a numerical example of a leaking pipe where the pipe length is $l = 2,000$ m, the internal diameter is $D = 0.3$ m, the wave speed is $a = 1,000$ m/s, and the leak location is $x_L = 1,500$ m. The change in leak size is represented by the parameter $Z_C / Z_L$. In theory, $Z_C / Z_L$ ranges from zero to infinity. Here, we use $Z_C / Z_L = 10^6$ in the numerical computation to approximate the infinitely large-sized leak; $Z_C / Z_L = 10^6$ is arbitrarily chosen but is sufficiently large in this numerical case to guarantee the convergence of normal-mode wavenumbers regarding the value of $Z_C / Z_L$ is justified.

The complex-valued normal-mode wavenumbers are calculated via Eq. (10) and shown in Fig. 2. The arrows in this figure show the trajectory of each normal-mode wavenumber in the complex plane with increasing leak size. The exact values of normal-mode wavenumbers for some specific $Z_C / Z_L$ are shown in Table 1. The complex-valued normal modes are normalized by the fundamental normal-mode wavenumber $k_{NL}^n$ of the intact pipeline and

![Fig. 2. Trajectories of normal modes in the complex plane, where $Z_C / Z_L = 0, 2, 5$ (correspondingly, $C_B A_l / A = 0\%, 3\%, and 8\%$) are labeled.](image)

### Table 1. Normalized normal modes for the illustrated pipeline case

<table>
<thead>
<tr>
<th>$Z_C / Z_L$</th>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
<th>Mode 4</th>
<th>Mode 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>1.000 + 0.000i</td>
<td>3.000 + 0.000i</td>
<td>NaN</td>
<td>5.000 + 0.000i</td>
<td>7.000 + 0.000i</td>
</tr>
<tr>
<td>0.27</td>
<td>0.998 + 0.074i</td>
<td>3.000 + 0.013i</td>
<td>NaN</td>
<td>5.000 + 0.013i</td>
<td>7.002 + 0.074i</td>
</tr>
<tr>
<td>1.00</td>
<td>0.974 + 0.279i</td>
<td>2.996 + 0.047i</td>
<td>NaN</td>
<td>5.004 + 0.047i</td>
<td>7.026 + 0.279i</td>
</tr>
<tr>
<td>1.45</td>
<td>0.942 + 0.415i</td>
<td>2.991 + 0.067i</td>
<td>NaN</td>
<td>5.009 + 0.067i</td>
<td>7.058 + 0.415i</td>
</tr>
<tr>
<td>2.00</td>
<td>0.871 + 0.602i</td>
<td>2.983 + 0.091i</td>
<td>NaN</td>
<td>5.017 + 0.092i</td>
<td>7.129 + 0.602i</td>
</tr>
<tr>
<td>2.40</td>
<td>0.773 + 0.772i</td>
<td>2.975 + 0.111i</td>
<td>NaN</td>
<td>5.023 + 0.111i</td>
<td>7.227 + 0.772i</td>
</tr>
<tr>
<td>3.00</td>
<td>0.398 + 1.102i</td>
<td>2.959 + 0.136i</td>
<td>NaN</td>
<td>5.041 + 0.136i</td>
<td>7.602 + 1.102i</td>
</tr>
<tr>
<td>3.13</td>
<td>0.073 + 1.194i</td>
<td>2.955 + 0.142i</td>
<td>NaN</td>
<td>5.045 + 0.142i</td>
<td>7.927 + 1.194i</td>
</tr>
<tr>
<td>4.00</td>
<td>0.000 + 0.534i</td>
<td>2.924 + 0.174i</td>
<td>NaN</td>
<td>5.076 + 0.174i</td>
<td>8.000 + 0.534i</td>
</tr>
<tr>
<td>4.20</td>
<td>0.000 + 0.494i</td>
<td>2.915 + 0.180i</td>
<td>4.000 + 3.874i</td>
<td>5.084 + 0.180i</td>
<td>8.000 + 0.494i</td>
</tr>
<tr>
<td>5.00</td>
<td>0.000 + 0.386i</td>
<td>2.878 + 0.198i</td>
<td>4.000 + 2.015i</td>
<td>5.122 + 0.198i</td>
<td>8.000 + 0.386i</td>
</tr>
<tr>
<td>6.00</td>
<td>0.000 + 0.307i</td>
<td>2.831 + 0.205i</td>
<td>4.000 + 1.332i</td>
<td>5.169 + 0.205i</td>
<td>8.000 + 0.308i</td>
</tr>
<tr>
<td>12.00</td>
<td>0.000 + 0.144i</td>
<td>2.709 + 0.135i</td>
<td>4.000 + 0.468i</td>
<td>5.291 + 0.135i</td>
<td>8.000 + 0.144i</td>
</tr>
<tr>
<td>20.00</td>
<td>0.000 + 0.085i</td>
<td>2.682 + 0.084i</td>
<td>4.000 + 0.263i</td>
<td>5.318 + 0.084i</td>
<td>8.000 + 0.085i</td>
</tr>
<tr>
<td>1.00 × 10^6</td>
<td>0.000 + 0.000i</td>
<td>2.667 + 0.000i</td>
<td>4.000 + 0.000i</td>
<td>5.333 + 0.000i</td>
<td>8.000 + 0.000i</td>
</tr>
</tbody>
</table>

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denoted by $k^*$. The value $[\text{Re}(k^*), \text{Im}(k^*)]$ is used to represent the coordinate of a normal-mode wavenumber in the complex plane. It is seen from Fig. 2 that $k^*$ is periodic along the real axis with a period $\text{Re}(k^*) = 8$. In what follows, we consider three periods (i.e., $\text{Re}(k^*) \in [0, 24]$).

When there is no leak ($Z_C/Z_L = 0$), all the normalized normal-mode wavenumbers are located on the real axis at the odd harmonics ($k_n^* = 2n - 1$, $n \in \mathbb{N}^+$). In the presence of a leak ($Z_C/Z_L \neq 0$), they shift from the odd harmonics not only along the real axis, but also toward the upper half of the complex plane. This fact indicates the existence of the leak-induced shift of normal modes. In fact, as $Z_C/Z_L$ varies from 0 to about 4, Fig. 2 and Table 1 show that the normalized normal-mode wavenumbers that start at normalized normal-mode wavenumbers $\text{Re}(k^*) = 1$ and $\text{Re}(k^*) = 7$ shift more than the normalized normal modes that start at normalized normal-mode wavenumbers $\text{Re}(k^*) = 3$ and $\text{Re}(k^*) = 5$ on both the real and imaginary axes. Also, the shift in the imaginary axis is much more significant than the shift in the real axis, indicating that the considered leak influences the damping more than the oscillatory frequency.

When the leak size increases to approximately $Z_C/Z_L = 4.2$, new dissipative modes (referred to as Mode 1* in Table 1) appear at $\text{Re}(k^*) = 8n - 4$, $n \in \mathbb{N}^+$. The trajectories of these new-mode wavenumbers are toward the real axis as the leak size increases (see arrows in Fig. 2 modes 4, 12, and 20). The physical origin of the new normal modes is theoretically explained in the next subsection.

As $Z_C/Z_L$ further increases up to about 20, which implies a very large leak size, as shown in Fig. 2, the trajectories that start at normalized normal-mode wavenumbers $\text{Re}(k^*) = 7$ and $\text{Re}(k^*) = 9$ merge into $\text{Re}(k^*) = 8$. Similarly, the normal-mode wavenumbers $\text{Re}(k^*) = 15$ and $\text{Re}(k^*) = 17$ finally merge into $\text{Re}(k^*) = 16$.

As $Z_C/Z_L$ tends to infinity, the system only has normalized normal-mode wavenumbers at real-valued normalized normal-mode wavenumbers $k_n^* = \text{Re}(k_n^*) = 8n - 4$, $n \in \mathbb{N}^+$ and $k_n^* = \text{Re}(k_n^*) = 8n/3$. In fact, $\text{Re}(k_2^*) = 8n - 4$ and $\text{Re}(k_3^*) = 8n/3$ are, respectively, the normalized normal-mode wavenumbers for the pipe sections with lengths $l - x_L = 500$ and $x_L = 1,500$ m. The bifurcation of the pipe into two length scales is discussed in the next subsection.

**Case with Extremely Large Leak**

The physical meaning of this limiting case of leak size is illustrated here. When the leak becomes extremely large, $Z_C/Z_L \gg 1$, the first term of Eq. (7) can be neglected and thus Eq. (7) becomes

$$f(k) = \frac{Z_C}{2Z_L} \cos(k(l - x_L)) \sin(kx_L) = 0 \quad (12)$$

The normal modes satisfying Eq. (12) let either

$$\cos(k(l - x_L)) = 0 \quad (13)$$

or

$$\sin(kx_L) = 0 \quad (14)$$

The corresponding normal modes are

$$k^{l-s_1} = \frac{2n - 1}{2(l - x_L)}, \quad n \in \mathbb{N}^+ \quad (15)$$

and

$$k^{x_L} = \frac{n \pi}{x_L}, \quad n \in \mathbb{N}^+ \quad (16)$$

Their normalized forms are

$$\left(\frac{(k^{l-s_1})^*}{k_1^{NL}}\right) = \frac{(2n - 1) \frac{\pi}{2(l - x_L)}}{\frac{\pi}{2(2n - 1)}} = 4(2n - 1), \quad n \in \mathbb{N}^+ \quad (17)$$

and

$$\left(\frac{(k^{x_L})^*}{k_1^{NL}}\right) = \frac{n \frac{\pi}{x_L}}{\frac{\pi}{x_L}} = \frac{8}{3} n, \quad n \in \mathbb{N}^+ \quad (18)$$

respectively. As a matter of fact, when the leak size becomes extremely large, it is more capable of holding the local pressure head constant at $H_{atm}$ and can be regarded as a reservoir (Ferrante et al. 2001; Zecchin et al. 2008), as shown in Fig. 3. In this situation, the pipeline can be considered as being divided into two separate individual intact pipe sections. One section extends from the upstream reservoir to the leak, has length $x_L = 1,500$ m, and can be regarded as a reservoir-pipe-valve (RPV) system. The other section extends from the leak to the downstream valve and can be regarded as a reservoir-pipe-reservoir (RPR) system. The normalized normal modes in Eqs. (17) and (18) correspond to the RPR system and the RPV system, respectively. It can be seen from Fig. 2 and Table 1 that when the leak size becomes extremely large, the normal modes are composed of the

![Fig. 3. Layout of a pipeline with an extremely large leak.](image-url)
Fig. 4. Relationship between normal modes and FRF\^D. The pipe length is \( l = 2,000 \text{ m} \), the internal diameter is \( D = 0.3 \text{ m} \), the wave speed is \( a = 1,000 \text{ m/s} \), the leak is located at \( x_L = 1,500 \text{ m} \), and the leak-related parameter \( Z_C/Z_L = 0.5 \). (a) The first normal mode; (b) the second normal mode; (c) plan view of (a) for a clearer look at the peak location; (d) plan view of (b) for a clearer look at the peak location; and (e) FRF\^D (with real-valued wavenumber).
two groups of normal modes in Eqs. (17) and (18), each of which corresponds to each subsection of pipe.

**Relationship between Normal Modes and FRF (Inhomogeneous Solution)**

**Frequency Response Function**

The analysis performed in the last section is for the homogeneous wave problem (i.e., no forcing is applied). In this section, the forced wave problem is investigated. In practice, the forcing is often in the form of a flow or pressure disturbances. Consider flow perturbation \( q^D \). Therefore, \( h^D \) is the particular inhomogeneous solution due to \( q^D \) forcing. The pressure head response per unit flow perturbation at the downstream, which is also known as FRF\(^D\), can be derived from Eqs. (5) and (6) as follows:

\[
\text{FRF}^D = \frac{h^D}{q^D} = \frac{-i Z_C \sin(kl) + \frac{Z_C}{2 \cos(kl)} \sin(k(l - x_L)) \sin(kx_L)}{\cos(kl) + \frac{Z_C}{2 \cos(kl)} \sin(k(l - x_L)) \sin(kx_L)}
\]  

(19)

where \( k = \omega/a \); and \( \omega = \) forcing frequency. This forcing frequency by, say, a valve maneuver is real valued. It is important to emphasize here that the homogeneous solution admits complex-valued \( k \); the particular solution admits real-valued \( k \) only because both the forcing frequency \( \omega \) and the wave speed are real. The set of real-valued \( k = \omega/a \), where \(|h^D|/|q^D|\) is the maximum are often referred to as resonant frequencies (Lee et al. 2002; Mpesha et al. 2001). It is important to emphasize that, strictly, this is a misnomer because for the case of a pipe with a leak, such a maximum \(|h^D|/|q^D|\) finite due to damping by the leak. This fact is illustrated by an example (see Fig. 4, where the first and seconds modes are shown). In this example, the pipe setup is the same as the one in the previous section, with the leak-related parameter \( Z_C/Z_L = 0.5 \). In Figs. 4(a–d), \( 1/f(k) \), where \( f(k) \) is obtained from the homogeneous wave problem, is plotted in the complex plane of \( k \), which is approximately symmetric in both real and imaginary axes around the center of each resonant wavenumber. The value \( 1/f(k) \) illustrates the resonance peaks in the complex domain. The real part of the complex-valued normal-mode wavenumber, where \( 1/f(k) \) goes to infinity, is approximately equal to the location of FRF\(^D\)'s peak in Fig. 4(e). The shift of the first normal mode along the real axis from the corresponding normal mode of the intact pipe (the dashed line in Fig. 4(c)] is greater than the shift in the second normal mode [Fig. 4(d)]. As a result, the first mode experiences more leak-induced shift of the FRF's peak than the second mode [see FRF\(^D\) in Fig. 4(e)]. Similarly, the shift of the first normal mode along the imaginary axis is greater than the second normal mode. Therefore, the damping of the first peak in FRF\(^D\) in Fig. 4 is larger than the second peak.

**Variation of FRF due to Leak**

In this subsection, the variation of the FRF and especially its peaks as a function of leak size are studied. The pipeline setup is as in the previous sections. The change of the FRF due to variations in leak is shown in Fig. 5. In this figure, six curves with different leak sizes are plotted (Fig. 2).

When there is no leak \( (Z_C/Z_L = 0) \), as shown in Table 1 and Fig. 5, all the normal-mode wavenumbers are located at the real axis at the odd harmonics \([\Re(k_n^+)=2n−1, n \in \mathbb{N}^+]\), where FRF\(^D\) goes to infinity. For \( Z_C/Z_L = 0.3 \) and \( Z_C/Z_L = 4 \), the shift in wavenumber where FRF\(^D\) is maximum is small and can be neglected, as reported by Lee (2005). However, as \( Z_C/Z_L \) increases to 12, as can be observed from line (3) in Fig. 5, the value of FRF\(^D\) gradually decreases at the original normal-mode wavenumbers (odd harmonics), but increases at \( \Re(k^+) = 4, 12, 20 \). Furthermore, the shift of the peak location of FRF\(^D\) can be clearly observed. As \( Z_C/Z_L \) increases to 20, the peaks and zeros of FRF\(^D\) are at the same position as those of \( Z_C/Z_L = \infty \). As discussed in the last section, when the leak is large, its behavior becomes similar to a reservoir. That is, the pipe system is divided into two subsystems: one subsystem extends from the upstream reservoir to the leak and the other extends from the leak to the downstream valve. Generating the transient at the valve and investigating the response FRF\(^D\) at the valve means as the leak becomes large, one picks up the wave dynamics that occur in the pipe section that extends from the leak to the valve only. That is, the wave transmission at the leak toward the subsystem that extends from the reservoir to the leak becomes negligible as the leak gets larger. To illustrate this fact, consider the FRF at a point \( x < x_L \) due to a transient generated at \( x = l \). The complex head \( h^s \) and discharge \( q^s \) are obtained from the transfer matrix equation

\[
\begin{bmatrix}
q^U \\
h^U
\end{bmatrix} = 
\begin{bmatrix}
\cos(kx) & -\frac{1}{Z_C}\sin(kx) \\
-iZ_C\sin(kx) & \cos(kx)
\end{bmatrix}
\begin{bmatrix}
q^U \\
h^U
\end{bmatrix}
\]  

(20)

Given the boundary condition \( h^U = 0 \), we have

\[
\text{FRF}^s = \frac{h^s}{q^s} = -i Z_C \sin(kx) q^U / q^s
\]  

(21)

by Eq. (5), it becomes
Brunone (2003), and Mpesha et al. (2002) are revisited. The system parameters in Lee et al. (2005b) are as follows. The wave speed is

\[ a = 1,200 \text{ m/s}, \]  

the dimensionless leak size \( C_d A_L / A \) is less than \( 2 \times 10^{-3} \), and the head at the leak is greater than 50 m, which leads to the leak-related parameter \( Z_C / Z_L = (\alpha Q_L) / (g AH_L) \) being less than 0.1. In this case, according to our computation by LRBQM, the maximum normalized resonant wavenumber shift, compared to an intact pipe, in real and imaginary parts is \( \text{Re}(\Delta k^*) < 0.025\% \) and \( \text{Im}(\Delta k^*) < 3\% \), respectively. This result implies that the shift of the real part of the frequency is small and can be neglected. The emergence of the imaginary part of the frequency is a measure of the leak-induced damping. These results corroborate the results of Ferrante et al. (2003), where the pipe is made of polyethylene, with \( l = 350.5 \text{ m}, D = 93.8 \text{ mm}, \) and \( a = 330 \text{ m/s}. \) One of the experimental tests was carried out for a leak parameter \( Z_C / Z_L = 2.4. \) A noticeable shift of the resonant wavenumber was found [see Figs. 5 and 6 in Ferrante and Brunone (2003)]. Using the LRBQM-based computation proposed in the present paper, the maximum normalized resonant wavenumber shift is \( \text{Re}(\Delta k^*) \approx 25\% \) and \( \text{Im}(\Delta k^*) \approx 8\% \) when \( Z_C / Z_L = 2.4. \) This means that the shift of the resonant wavenumber along the real axis is indeed large. However, no new resonant wavenumbers appear.

Another numerical test considered in Ferrante and Brunone (2003) was for \( Z_C / Z_L = 20. \) This is an extremely large leak case and the transmission at the leak of a wave that is generated at the

\[ \sin(k(l - x_L)) = 0 \Rightarrow k^{l-x_L} (l - x_L) = n\pi \Rightarrow (k^{l-x_L})^* = 8n, \quad n \in \mathbb{N}^+ \]  

Therefore, the normalized antinormal modes of this section are \( (k^{l-x_L})^* = 4(2n - 1) = 8, 16, 24, 32, \) and so on. In fact, \( \text{Frf}^D \) is at its minimum for \( (k^{l-x_L})^* = 8, 16, 24 \) (as shown in Fig. 5) for \( Z_C / Z_L = 20 \) and \( Z_C / Z_L = \infty. \)

**Revisiting the Examples in the Literature**

The examples presented in Lee et al. (2005b), Ferrante and Brunone (2003), and Mpesha et al. (2002) are revisited. The system parameters in Lee et al. (2005b) are as follows. The wave speed is

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Another numerical test considered in Ferrante and Brunone (2003) was for \( Z_C / Z_L = 20. \) This is an extremely large leak case and the transmission at the leak of a wave that is generated at the
valve is negligible. Therefore, the FRF at the valve is mainly that of the pipe subsystem that extends from the leak to the valve. The results in Ferrante and Brunone (2003) are consistent with the theory in the present paper.

One numerical example in Mpesha et al. (2002) was for $Z_C/Z_L = 0.11$. In this case, the current analysis shows that the largest frequency shift is $Re(\Delta f^*) = 0.12\%$, which is so small that it can be neglected. Very importantly, no new frequencies appear. However, for the case with $Z_C/Z_L = 2.7$, new frequencies do appear and represent the pipe subsystem that extends from the leak to the downstream valve.

The previous results prove that each of the hypotheses regarding the impact of a leak on the FRF advanced in the literature is only valid for a certain range of $Z_C/Z_L$. The assumption of no shift and no new peak in Lee et al. (2005b) is valid when the leak is small ($Z_C/Z_L$ of order 1 or less). On the other hand, when the leak is large (say, $Z_C/Z_L \gg 1$), the normal-mode shift is not negligible and the change of the FRF’s shape is significant; that is, the original peaks disappear and new peaks appear, which is in accord with the work of Ferrante and Brunone (2003) and Mpesha et al. (2002). One of the authors of Mpesha et al. (2002) commented in the reply to Lee et al. (2010) that “a leakage from a piping system constitutes an additional boundary and should result in changing the frequency response of the system.” The analysis conducted in this paper leads to the following revised version of this statement: a leakage from a piping system constitutes an additional boundary and results in significant change in the frequency response, including the appearance of new frequencies, of the system when $Z_C/Z_L$ is large.

### Alternative Method for Computing Leak-Induced Normal-Mode Shift

Even though the leak-induced normal-mode shift can be numerically calculated by LRBQM in Eqs. (10) and (11), the computational speed is low. This is due to numerical integration as well as the need for a careful choice of the integration contour $C_n$. The requirement is that $C_n$ encircle only one singularity $[W = 1$ in Eq. (31)], and this requires trial and error. Therefore, a simpler and faster method for computing leak-induced normal-mode shift is desired.

### Methodology

We propose an explicit but approximate solution of the leak-induced normal-mode shift based on a first-order Taylor expansion. This is motivated by the fact that, in practice, the leaks that are difficult to identify and require advanced methodologies to find are the small ones. These small leaks are characterized by small $Z_C/Z_L$. Leaks with large $Z_C/Z_L$ are easily detected and do not need advanced identification schemes.

Let

$$k_n^L = k_n^L + \Delta k_n = (2n-1) \frac{\pi}{2L} + \Delta k_n, \quad n \in \mathbb{N}^+ \quad (24)$$

where $k_n^L$ is wavenumber with a leak; $k_n^NL$ is wavenumber with no leak; and $\Delta k_n$ is wavenumber shift due to a leak. Eq. (24) implicitly assumes a one-to-one mapping from the normal modes in the intact pipe to those in the leaking pipe: the number of normal modes remains and the shift is continuous, which is a reasonable assumption for small leaks according to the results in the previous section. Expanding $f(k_n^L)$ at $k_n^NL$ using a Taylor series and keeping the first-order term only gives

$$f(k_n^L) = f(k_n^NL + \Delta k_n) \approx f(k_n^NL) + \Delta k_n f'(k_n^NL) = 0 \quad (25)$$

thus, the expression of the shift $\Delta k_n$ is

$$\Delta k_n = -f'(k_n^NL) = \frac{\frac{4}{3} i \omega Z_C Z_L \sin^2(k_n^NL x_L) \cos(k_n^NL x_L)}{\frac{4}{3} \omega Z_C Z_L \sin^2(k_n^NL x_L) \cos(k_n^NL x_L) + \frac{4}{3} (1 - 2 x_L) \sin^2(k_n^NL x_L) \cos^2(k_n^NL x_L)} \quad (26)$$

The derivation of Eq. (26) is detailed in Appendix II. The real and imaginary parts of Eq. (26) are, respectively

$$Re(\Delta k_n) = \frac{\frac{2}{3} \omega Z_C Z_L \sin^2(k_n^NL x_L)}{\frac{4}{3} \omega Z_C Z_L \sin^2(k_n^NL x_L) + \frac{4}{3} (1 - 2 x_L) \sin^2(k_n^NL x_L) \cos^2(k_n^NL x_L)} \quad (27)$$

$$Im(\Delta k_n) = \frac{\frac{2}{3} \omega Z_C Z_L \sin^2(k_n^NL x_L)}{\frac{4}{3} \omega Z_C Z_L \sin^2(k_n^NL x_L) + \frac{4}{3} (1 - 2 x_L) \sin^2(k_n^NL x_L) \cos^2(k_n^NL x_L)} \quad (28)$$

The normal mode for a leaking pipeline is obtained as $k_n^L = k_n^NL + \Delta k_n$: $Re(k_n^L) = k_n^NL + Re(\Delta k_n)$; $Im(k_n^L) = Im(\Delta k_n)$ (29)

The accuracy of the approximate solution is verified in the following subsection.

### Numerical Verification

Numerical examples are used to verify Eqs. (26)–(28). The pipeline system is as before. The normalized normal-mode shift $\Delta k_n^\alpha = \Delta k_n/k_n^NL$ computed from Eq. (26) is plotted as the solid line in Fig. 7 and is compared with the “exact” solution obtained from LRBQM in Eq. (11), which is shown by the dashed lines. Two leak sizes with different $Z_C/Z_L = 0.3$ and 1.5 are considered. These two choices of $Z_C/Z_L$ are realistic; the value of $Z_C/Z_L$ is in general smaller than 2 in practice (Ferrante and Brunone 2003). For example, if $H_L$ is within the range 30–60 m, the wave speed is of the order of 1000 m/s, and let $C_p A_L/\omega$ belong to the range 0.001–0.01. The corresponding $Z_C/Z_L$ is in the range 0.061-1.

In the case of a small leak size with $Z_C/Z_L = 0.3$, Figs. 7(a and b) show that both the real and imaginary parts of the normalized shift $\Delta k_n^\alpha$ perfectly coincide with those obtained from the “precise” solution using LRBQM. This means that when the leak size is small, Eqs. (26)–(28) accurately describe the shift in normal mode. For the relatively large leak with $Z_C/Z_L = 1.5$, it is clear from Figs. 7(c and d) that the shift given by the explicit expression is slightly different from that obtained by LRBQM, which means that in the case of large leaks, the first-order Taylor expansion approximation is less accurate, but still acceptable.

The relative error in Eq. (26) for different $Z_C/Z_L$ is plotted in Fig. 8. The relative error of the real part (the imaginary part and modulus can be similarly defined) of the shift is defined as

$$E^\alpha = \frac{1}{N} \sum_{i=1}^{N} \left| Re(\hat{\Delta} k_n^\alpha) - Re(\Delta k_n^\alpha) \right| \times 100\% \quad (30)$$

where $\hat{\Delta} k_n^\alpha$ denotes the normalized resonant wavenumber shift calculated by the explicit expression Eq. (26); $\Delta k_n^\alpha$ denotes the actual shift, which is in fact computed from LRBQM; and $N$ = number of points within one period. In this case, it can be seen from Fig. 7 that $N = 4$. The relative error of the norm (solid line), imaginary part (dashed line), and real part (dotted line) of the shift are all plotted in Fig. 8. As the leak size increases, the error of the explicit expression...
increases, whereby the main difference of the shift estimation comes from the imaginary part. However, within the practical range $0 < Z_C/Z_L < 1$, the relative error is less than 4.5%, which is more than acceptable.

**Conclusion**

This paper is prompted by the controversial issue in the literature as to whether a leak induces frequency peaks that are not present in the intact pipe system. To systematically address this issue, both the homogeneous and particular solutions for a simple pipe system with a leak are analyzed theoretically. It is found that the frequency shift is negligible and no new peaks appear when the leak is small ($Z_C/Z_L$ of order 1 or less). On the other hand, when the leak is large (say, $Z_C/Z_L \gg 1$), the frequency is no longer negligible and the change of the FRF’s shape is significant; that is, the original peaks disappear and new peaks appear. Physically, small leaks provide damping. Large leaks, on the other hand, behave as a reservoir. It is concluded that the controversy in the literature is due to the fact that different authors were investigating vastly different leak sizes. Therefore, it is natural that they arrived at different conclusions.

**Appendix I. Logarithmic Residue-Based Quadrature Method**

If a complex function $f(z)$ and its derivative $f'(z)$ are analytic in a simply connected closed set $C$, the number of zeros of $f$ within $C$ can be obtained using trapezoidal rule approximations to the
contour integral (Delves and Lyness 1967; Kravanja and Van Barel 2007)

\[ W = \frac{1}{2\pi i} \int_{\delta C} \frac{f'(z)}{f(z)} \, dz \]  (31)

where \( \delta C \) = boundary of \( C \).

Let \( Z_i, i = 1, \ldots, W \), denote the zeros in \( C \); then, the \( N \)th order power sum of the zeros is defined by

\[ s_N = \sum_{i=1}^{W} (z_i^N), \quad N \in \{0, 1, \ldots, W\} \]  (32)

which can be obtained using the trapezoidal rule

\[ s_N = \frac{1}{2\pi i} \int_{\delta C} z^N \frac{f'(z)}{f(z)} \, dz \]  (33)

If there is only one zero inside \( C \); that is, \( W = 1 \), then this zero (denoted by \( z_1 \)) can be computed by the first-order power

\[ z_1 = s_1 = \frac{1}{2\pi i} \int_{\delta C} z f'(z) \, dz \]  (34)

Appendix II. Explicit Expression for Leak-Induced Resonant Wavenumber Shift

Eq. (7) is repeated here as

\[ f(k) = \cos(kl) + i \frac{Z_C}{2Z_L} \cos(k(l - x_L)) \sin(kx_L) \]  (35)

and its derivative

\[ f'(k) = -i \sin(kl) + i \frac{x_L Z_C}{2Z_L} \cos(kx_L) \cos(k(l - x_L)) \]

\[ - i \frac{l - x_L}{2} \frac{Z_C}{Z_L} \sin(kx_L) \sin(k(l - x_L)) \]  (36)

Therefore,

\[ f(k_{NL}) = \frac{1}{2} (-1)^{n+1} \frac{Z_C}{Z_L} \sin^2(k_{NL} x_L) \]  (37)

\[ f'(k_{NL}) = -i \sin(k_{NL} l) + i \frac{x_L Z_C}{2Z_L} \cos(k_{NL} x_L) \cos(k_{NL}(l - x_L)) \]

\[ - i \frac{l - x_L}{2Z_L} \sin(k_{NL} x_L) \sin(k_{NL}(l - x_L)) \]  (38)

By substituting Eqs. (37) and (38) into Eq. (25), the expression of the leak-induced resonant wavenumber shift is

\[ \Delta k_n = - \frac{f(k_{NL})}{f'(k_{NL})} = \frac{\frac{1}{2} i \frac{Z_C}{Z_L} \sin^2(k_{NL} x_L)}{l + \frac{1}{2} i(l - 2x_L) \frac{Z_C}{Z_L} \sin(k_{NL} x_L) \cos(k_{NL} x_L)} \]  (39)

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References


